

MATRICIALLY FREE RANDOM VARIABLES

ROMUALD LENCZEWSKI

ABSTRACT. We show that the operatorial framework developed by Voiculescu for free random variables can be extended to arrays of random variables whose multiplication imitates matricial multiplication. The associated notion of independence, called *matricial freeness*, can be viewed as a generalization of both freeness and monotone independence. At the same time, the sums of matricially free random variables, called *random pseudomatrices*, are closely related to Gaussian random matrices. The main results presented in this paper concern the standard and tracial central limit theorems for random pseudomatrices and the corresponding limit distributions which can be viewed as matricial generalizations of semicircle laws.

1. INTRODUCTION

It has been shown by Voiculescu [22] that free random variables arise naturally as limits of random matrices. In particular, if we take symmetric matrices whose entries form a family of independent Gaussian random variables and we let the size of these matrices go to infinity, their moments (with respect to normalized trace composed with classical expectation) converge to the moments of freely independent random variables with the semicircle distribution obtained by Wigner [25] as the limit distribution of one Gaussian random matrix.

Therefore, we can study free random variables, using at least two different frameworks: operator algebras and random matrices. However, it is to some extent surprising that a random matrix framework, of quite different nature than that of operator algebras, exists for free random variables, and the connection between these two approaches does not seem to be very transparent. In this connection, our first motivation is to better understand the relation between the operatorial approach to free probability and random matrices.

The second motivation comes from the question whether different types of independence, like freeness and monotone independence of Muraki [16], can be included in one natural model. Note in this context that models with more than one state on a given algebra, like conditional freeness of Bożejko and Speicher [6] and freeness with infinitely many states of Cabanal-Duivillard and Ionesco [9,10] extend free probability and include, as shown by Franz [11], certain elements of monotone probability. For instance, this can be done for convolutions, but including monotone independence in the framework of conditional freeness can be done only under additional (rather restrictive)

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assumptions on the considered algebras. We show in this paper that one can remedy this situation by introducing a concept of ‘independence’ which reminds freeness, but at the same time has some ‘matricial’ features which places it somewhere between freeness and the model of random matrices.

A different reason to look for a new concept of independence arises from concrete examples of interpolations between free probability and monotone probability [14,15]. In particular, the continuous (p,q) -Brownian motions with Kesten distributions and related Poisson processes studied in [2,15] lead to the first example of a two-mode interacting Fock space introduced directly and not by means of orthogonal polynomials. This example has certain ‘matricial’ features which also call for a new model of ‘independence’ that would be related to freeness.

The main result of our paper is the construction of a model, called *matricial freeness*, which is related to the concept of the free product of states introduced and studied by Ching [10] in the context of von Neumann algebras and by Avitzour [3] and Voiculescu [20] in the context of C^* -algebras. The underlying concept is that of the matricially free product of an array of Hilbert spaces with distinguished unit vectors $(\mathcal{H}_{i,j}, \xi_{i,j})$ which reminds the free product of Hilbert spaces. However, it also satisfies the condition imitating matrix multiplication and the condition of ‘diagonal subordination’ which says that tensor products must end with ‘diagonal’ Hilbert spaces. Similarities between free probability and ‘matricially free probability’ hold also on other levels, some of which are studied in this paper. Strictly speaking, however, one needs to take a restriction of the matricially free product of states, called the *strongly matricially free product* of states, to recover the free product as a special case.

Roughly speaking, if $(X_{i,j}(n))_{1 \leq i,j \leq n}$ is an array of self-adjoint matricially free random variables in a $*$ -algebra \mathcal{A}_n , equipped with a distinguished state ϕ_n and a sequence $(\psi_{n,j})_{1 \leq j \leq n}$ of additional states, called ‘conditions’, for each natural n , we can take the sums

$$S(n) = \sum_{i,j=1}^n X_{i,j}(n)$$

called *random pseudomatrices*, and study their asymptotic distributions (as $n \rightarrow \infty$) with respect to the states ϕ_n and with respect to normalized traces

$$\psi_n = \frac{1}{n} \sum_{j=1}^n \psi_{n,j},$$

respectively. We assume that the distributions of the $X_{i,j}(n)$ in the states ϕ_n and $\psi_{n,k}$ are not identical and depend on n in such a way that their ψ_n -distributions remind those of Gaussian random matrices in the approach of Voiculescu. This relation to random matrices as well as asymptotic matricial freeness of random pseudomatrices will be studied in a forthcoming paper. In progress is the work on the addition of matricially free random variables and the resulting matricial R-transform.

It turns out that the limit theorem for the above random pseudomatrices in the states ϕ_n , which we call ‘standard’, may be viewed as an analog of the central limit theorem for free random variables (especially, if we take square arrays). In turn, the limit theorem for random pseudomatrices in the states ψ_n , called ‘tracial’, is related

to the limit theorem for random matrices (especially, if we take square arrays). The limit distributions play then the role of multivariate generalizations of the semicircle distributions. Let us point out, however, that when we consider triangular arrays, the framework of matricial freeness can as well be viewed as a generalization of that of monotone independence.

In Section 2, we introduce the concepts of the ‘matricially free product of states’ and the ‘matricially free Fock space’. We obtain from these structures their strong counterparts in Section 3. In Section 4, we introduce the notions of ‘matricial freeness’ and ‘strong matricial freeness’ and discuss the example of the discrete (strongly) matricially free Fock space. In Section 5, of combinatorial nature, we define and study certain real-valued functions on the set of non-crossing partitions, defined in terms of traces of certain matrices. In Section 6, we study the asymptotic behavior of random pseudomatrices and we prove standard and tracial central limit theorems. The limit distributions, which can be interpreted as matricial multivariate generalizations of semicircle laws, are studied in Section 7. Their decompositions in terms of s -free additive convolutions in the case of two-dimensional arrays are proved in Section 8. Two geometric realizations of the limit distributions, in terms of walks on weighted binary trees and in terms of weighted Catalan paths, are given in Section 9.

2. MATRICIALLY FREE PRODUCTS

In this Section we introduce the notion of the matricially free product of states as well as the corresponding notions of the matricially free product of Hilbert spaces and the matricially free Fock space.

When speaking of arrays indexed by two indices, say i, j , we shall usually assume that $i, j \in I$, where I is an index set. This refers to the situation when we deal with square arrays. However, we also want to consider other arrays, like triangular arrays of the form $T := \{(k, l) : k \geq l, ; k, l \in I\}$, where I is a linearly ordered index set. Therefore, by an *array* we will understand a subarray of a square array which *includes the diagonal*. Without loss of generality we can use the square array formulation most of the time. Of special interest will be the finite-dimensional case when $I = [n] := \{1, 2, \dots, n\}$.

Definition 2.1. Let $\hat{\mathcal{H}} := (\mathcal{H}_{i,j})$ be an array of complex Hilbert spaces. By the *matricially free Fock space over $\hat{\mathcal{H}}$* we understand the Hilbert space direct sum

$$\mathcal{M}(\hat{\mathcal{H}}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{(i_1, i_2) \neq \dots \neq (i_m, i_m) \\ n_1, \dots, n_m \in \mathbb{N}}} \mathcal{H}_{i_1, i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2, i_3}^{\otimes n_2} \otimes \dots \otimes \mathcal{H}_{i_m, i_m}^{\otimes n_m}$$

where Ω is a unit vector, with the canonical inner product.

Let us observe that the Hilbert space tensor powers which appear in the above direct sum have the following three properties:

- (1) the ‘matricial property’ – the second index of the preceding power agrees with the first index of the following power,
- (2) the ‘freeness property’ – the consecutive pairs of indices are different,
- (3) the ‘diagonal subordination property’ – the last pair is ‘diagonal’.

Of course, if the index set I consists of one element and thus $\hat{\mathcal{H}}$ is just one Hilbert space \mathcal{H} , the corresponding matricially free Fock space reduces to the usual free Fock space $\mathcal{F}(\mathcal{H})$. In general, however, $\mathcal{M}(\hat{\mathcal{H}})$ is a (usually, proper) subspace of the free Fock space $\mathcal{F}(\bigoplus_{i,j} \mathcal{H}_{i,j}) \cong {}^*_{i,j} \mathcal{F}(\mathcal{H}_{i,j})$.

Related to the ‘matricially free Fock space’ is the ‘matricially free product of Hilbert spaces’. The terminology parallels that introduced in free probability [20,23].

Definition 2.2. Let $(\mathcal{H}_{i,j}, \xi_{i,j})$ be an array of Hilbert spaces with distinguished unit vectors. By the *matricially free product* of $(\mathcal{H}_{i,j}, \xi_{i,j})$ we understand the pair (\mathcal{H}, ξ) , where

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{(i_1, i_2) \neq \dots \neq (i_m, i_m)} \mathcal{H}_{i_1, i_2}^0 \otimes \mathcal{H}_{i_2, i_3}^0 \otimes \dots \otimes \mathcal{H}_{i_m, i_m}^0,$$

with $\mathcal{H}_{i,j}^0 = \mathcal{H}_{i,j} \ominus \mathbb{C}\xi_{i,j}$ and ξ being a unit vector. We denote it $(\mathcal{H}, \xi) = {}^*_{{i,j}}^M(\mathcal{H}_{i,j}, \xi_{i,j})$.

Proposition 2.1. *It holds that*

$$(\mathcal{M}(\hat{\mathcal{H}}), \xi) \cong {}^*_{{i,j}}^M(\mathcal{M}(\mathcal{H}_{i,j}), \xi_{i,j})$$

Proof. We use the definition of the matricially free Fock space, the isomorphism $\mathcal{M}(\mathcal{H}_{i,j}) \cong \mathcal{F}(\mathcal{H}_{i,j})$ for any i, j and regroup terms. ■

For any j , introduce *diagonal subspaces* of \mathcal{H} of the form

$$\mathcal{H}(j, j) = \mathbb{C}\xi \oplus \bigoplus_{m=2}^{\infty} \bigoplus_{\substack{(j, i_2) \neq \dots \neq (i_m, i_m) \\ i_2 \neq j}} \mathcal{H}_{j, i_2}^0 \otimes \mathcal{H}_{i_2, i_3}^0 \otimes \dots \otimes \mathcal{H}_{i_m, i_m}^0,$$

and the associated *diagonal partial isometries* $V_{j,j} : \mathcal{H}_{j,j} \otimes \mathcal{H}(j, j) \rightarrow \mathcal{H}$:

$$\begin{aligned} \xi_{j,j} \otimes \xi &\rightarrow \xi \\ \mathcal{H}_{j,j}^0 \otimes \xi &\rightarrow \mathcal{H}_{j,j}^0 \\ \xi_{j,j} \otimes (\mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0) &\rightarrow \mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0 \\ \mathcal{H}_{j,j}^0 \otimes (\mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0) &\rightarrow \mathcal{H}_{j,j}^0 \otimes \mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0 \end{aligned}$$

where $m > 1$.

For any $i \neq j$ we introduce *non-diagonal subspaces* of \mathcal{H} of the form

$$\mathcal{H}(i, j) = \bigoplus_{m=1}^{\infty} \bigoplus_{(j, i_2) \neq \dots \neq (i_m, i_m)} \mathcal{H}_{j, i_2}^0 \otimes \mathcal{H}_{i_2, i_3}^0 \otimes \dots \otimes \mathcal{H}_{i_m, i_m}^0$$

and the associated *non-diagonal partial isometries* $V_{i,j} : \mathcal{H}_{i,j} \otimes \mathcal{H}(i, j) \rightarrow \mathcal{H}$ for $i \neq j$:

$$\begin{aligned} \xi_{i,j} \otimes (\mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0) &\rightarrow \mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0 \\ \mathcal{H}_{i,j}^0 \otimes (\mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0) &\rightarrow \mathcal{H}_{i,j}^0 \otimes \mathcal{H}_{j,j_1}^0 \otimes \dots \otimes \mathcal{H}_{j_m, j_m}^0, \end{aligned}$$

where $m \geq 1$.

Each $\mathcal{H}(i, j)$ is spanned by simple tensors which do not begin with vectors from $\mathcal{H}_{i,j}^0$ and for that reason it is suitable for the left free action of the operators creating such vectors. Thus, roughly speaking, both types of partial isometries jointly replace the

unitary maps used in free probability. It is the diagonal subordination property which is responsible for distinguishing two types of isometries.

Consider an array of C^* -algebras $(\mathcal{A}_{i,j})$, each with a unit $1_{i,j}$ and a state $\varphi_{i,j}$, and let $(\mathcal{H}_{i,j}, \pi_{i,j}, \xi_{i,j})$ be the associated GNS triples, so that $\varphi_{i,j}(a) = \langle \pi_{i,j}(a)\xi_{i,j}, \xi_{i,j} \rangle$ for any $a \in \mathcal{A}_{i,j}$. For any i, j , let $\lambda_{i,j}$ be the $*$ -representation of $\mathcal{A}_{i,j}$ given by

$$\lambda_{i,j}(a) = V_{i,j}(\pi_{i,j}(a) \otimes I_{\mathcal{H}(i,j)})V_{i,j}^* \text{ for } a \in \mathcal{A}_{i,j},$$

where $I_{\mathcal{H}(i,j)}$ denotes the identity on $\mathcal{H}(i,j)$. Note that these representations are, in general, non-unital. In fact,

$$\lambda_{i,j}(1_{i,j}) = V_{i,j}V_{i,j}^* = r_{i,j} + s_{i,j}$$

where $r_{i,j}$ and $s_{i,j}$ are canonical projections in $B(\mathcal{H})$ given by

$$r_{i,j} = P_{\mathcal{H}(i,j)} \text{ and } s_{i,j} = P_{\mathcal{K}(i,j)}$$

where $\mathcal{K}(i,j) = \mathcal{H}_{i,j}^0 \otimes \mathcal{H}(i,j)$. For given i, j , the projections $r_{i,j}$ and $s_{i,j}$ are orthogonal and their sum is the canonical projection onto the subspace of \mathcal{H} onto which $\lambda(\mathcal{A}_{i,j})$ acts non-trivially.

The $\lambda_{i,j}$ remind the representations λ_i of free probability, but the corresponding operators $\lambda_{i,j}(a)$ have larger kernels. Using $\lambda_{i,j}$'s, we shall define product representations on

$$\mathcal{A} := \sqcup_{i,j} \mathcal{A}_{i,j},$$

the free product without identification of units, equipped with the unit $1_{\mathcal{A}}$, and products of states which are analogs of the free product representation and the free product of states, respectively.

Definition 2.3. The *matricially free product representation* $\pi_M = *_{i,j}^M \pi_{i,j}$ is the unital $*$ -homomorphism $\lambda : \mathcal{A} \rightarrow B(\mathcal{H})$ given by the linear extension of

$$\lambda(1_{\mathcal{A}}) = \mathbf{1} \text{ and } \lambda(a_1 a_2 \dots a_n) = \lambda_{i_1, j_1}(a_1) \lambda_{i_2, j_2}(a_2) \dots \lambda_{i_n, j_n}(a_n)$$

for any $a_k \in \mathcal{A}_{i_k, j_k}$, $k = 1, \dots, n$, with $(i_1, j_1) \neq (i_2, j_2) \neq \dots \neq (i_n, j_n)$. The associated state $\varphi = *_{i,j}^M \varphi_{i,j} : \mathcal{A} \rightarrow \mathbb{C}$ is given by

$$\varphi(a) = \langle \pi_M(a)\xi, \xi \rangle$$

and will be called the *matricially free product of $(\varphi_{i,j})$* .

Basic properties of the product state φ are collected in the propositions given below. Roughly speaking, they show that this state (on the free product of C^* -algebras without identification of units) has similar properties as the free product of states (on the free product of C^* -algebras with identification of units) except that the units of these algebras act as units only on ‘matricial’ tensor products and otherwise they act as null projections.

For that purpose, it will be useful to introduce the following sets of indices:

$$\Lambda_n = \{((i_1, i_2), (i_2, i_3), \dots, (i_n, i_{n+1})) : (i_1, i_2) \neq (i_2, i_3) \neq \dots \neq (i_n, i_{n+1})\}$$

and $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$. Clearly, the set Λ encodes the matricial, freeness and diagonal subordination properties. Finally, \mathcal{I} stands for the unital subalgebra of \mathcal{A} generated by the units $1_{i,j}$.

Proposition 2.2. *Let φ be the matricially free product of states $(\varphi_{i,j})$ and let $a_k \in \mathcal{A}_{i_k,j_k}$, where $k \in [n]$ and $(i_1, j_1) \neq \dots \neq (i_n, j_n)$.*

- (1) *If $a_k \in \text{Ker } \varphi_{i_k,j_k}$ for $k \in [n]$, then $\varphi(a_1 a_2 \dots a_n) = 0$.*
- (2) *If $a_r = 1_{i_r,j_r}$ and $a_m \in \text{Ker } \varphi_{i_m,j_m}$ for $r < m \leq n$, then*

$$\varphi(a_1 \dots a_n) = \begin{cases} \varphi(a_1 \dots a_{r-1} a_{r+1} \dots a_n) & \text{if } ((i_r, j_r), \dots, (i_n, j_n)) \in \Lambda \\ 0 & \text{otherwise} \end{cases}.$$

- (3) *For any $a \in \mathcal{A}$, $u_1, u_2 \in \mathcal{I}$ and $i, j \in I$, it holds that*

$$\varphi(u_1 a u_2) = \varphi(u_1) \varphi(a) \varphi(u_2) \text{ and } \varphi(1_{i,j}) = \delta_{i,j}.$$

- (4) *The restriction of φ to $\mathcal{A}_{j,j}$ is $\varphi_{j,j}$ for any $j \in I$.*
- (5) *The mixed moments $\varphi(a_1 a_2 \dots a_n)$ are uniquely expressed in terms of mixed moments of products of a_k 's in the states φ_{i_k,j_k} .*

Proof. If $a_k \in \text{Ker } \varphi_{i_k,j_k}$ for $k \in [n]$, where $(i_1, j_1) \neq \dots \neq (i_n, j_n)$, then it follows from the definition of the $\lambda_{i,j}$ that

- (1) $\pi_M(a_1 \dots a_n) \xi = 0$ if $((i_1, j_1), \dots, (i_n, j_n)) \notin \Lambda$
- (2) $\pi_M(a_1 \dots a_n) \xi \in \mathcal{H}_{i_1,j_1}^0 \otimes \dots \otimes \mathcal{H}_{i_n,j_n}^0$ if $((i_1, j_1), \dots, (i_n, j_n)) \in \Lambda$.

In both cases we obtain a vector orthogonal to ξ on the RHS, which proves (1). Suppose now that the assumptions of (2) hold. If $((i_r, j_r), \dots, (i_n, j_n)) \in \Lambda$, then $\lambda_{i_r,j_r}(1_{i_r,j_r})$ acts as a unit on $\mathcal{H}_{i_{r+1},j_{r+1}}^0 \otimes \dots \otimes \mathcal{H}_{i_n,j_n}^0$ by the definition of the representations $\lambda_{i,j}$. On the other hand, $\lambda_{i_r,j_r}(1_{i_r,j_r})$ kills any simple tensor beginning with $h \in \mathcal{H}_{i_{r+1},j_{r+1}}^0$ if $j_r \neq i_{r+1}$ or $((i_{r+1}, j_{r+1}), \dots, (i_n, j_n)) \notin \Lambda$ since V_{i_r,j_r} does, which completes the proof of (2). In turn, (3) follows from the action of the $\lambda(1_{i,j})$ onto ξ . That φ agrees with $\varphi_{j,j}$ on $\mathcal{A}_{j,j}$ for any $j \in I$ follows from the action of the $\lambda_{j,j}(a)$, $a \in \mathcal{A}_{j,j}$, onto ξ , namely $\pi_M(a) \xi = (\pi_{j,j}(a) \xi)^0 + \varphi_{j,j}(a) \xi$, which gives (4). Finally, (5) is a consequence of (1)-(2). \blacksquare

In a similar way we can define states associated with other unit vectors from \mathcal{H} . For our purposes, we will need states associated with unit vectors $e_j \in \mathcal{H}_{j,j}^0$ which are in the ranges of $\pi_{j,j}(\mathcal{A}_{j,j})$, where $j \in I$, respectively, namely $\varphi_j : \mathcal{A} \rightarrow \mathbb{C}$ defined by the formulas

$$\varphi_j(a) = \langle \pi_M(a) e_j, e_j \rangle.$$

These states will be called *conditions* associated with φ and will be used for computing traces. They have similar properties as φ as the proposition given below shows in more detail. The main difference is that φ_j satisfies the condition of freeness type only for indices which satisfy $(j, j) \neq (i_1, j_1) \neq \dots \neq (i_n, j_n) \neq (j, j)$. Moreover, $\varphi_j|_{\mathcal{I}}$ is quite different than $\varphi|_{\mathcal{I}}$ due to different normalization conditions.

Proposition 2.3. *Let φ_j , where $j \in I$, be the conditions associated with $\varphi = \ast_{i,j}^M \varphi_{i,j}$ and let $a_k \in \mathcal{A}_{i_k,j_k}$, where $k \in [n]$ and $(j, j) \neq (i_1, j_1) \neq \dots \neq (i_n, j_n) \neq (j, j)$. Then*

- (1) *If $a_k \in \text{Ker } \varphi_{i_k,j_k}$ for $k \in [n]$, then $\varphi_j(a_1 a_2 \dots a_n) = 0$ for each j .*
- (2) *If $a_r = 1_{i_r,j_r}$ and $a_m \in \text{Ker } \varphi_{i_m,j_m}$ for $r < m \leq n$, then*

$$\varphi_j(a_1 \dots a_n) = \begin{cases} \varphi_j(a_1 \dots a_{r-1} a_{r+1} \dots a_n) & \text{if } ((i_r, j_r), \dots, (i_n, j_n)) \in \Lambda \\ 0 & \text{otherwise} \end{cases}.$$

(3) For any $a \in \mathcal{A}$, $u_1, u_2 \in \mathcal{I}$ and $i, j, k \in I$, it holds that

$$\varphi_j(u_1 a u_2) = \varphi_j(u_1) \varphi_j(a) \varphi_j(u_2) \text{ and } \varphi_j(1_{i,k}) = \delta_{j,k}.$$

(4) The restriction of φ_j to $\mathcal{A}_{i,j}$ is $\varphi_{i,j}$ for $i \neq j$.

(5) The mixed moments $\varphi_j(a_1 a_2 \dots a_n)$ are uniquely expressed in terms of mixed moments of products of a_k 's in the states φ_{i_k, j_k} .

Proof. The proof is similar to that of Proposition 2.2. The main difference concerns (3) and (4). In this context, notice that the unit vectors e_j play the same role with respect to the action of the $\lambda_{i,j}(a)$ for any $i \neq j$ as $\xi_{i,j}$ plays with respect to the action of $\pi_{i,j}(a)$, where $a \in \mathcal{A}_{i,j}$, and thus $\varphi_j(a) = \langle \lambda_{i,j}(a) e_j, e_j \rangle = \langle \pi_{i,j}(a) \xi_{i,j}, \xi_{i,j} \rangle = \varphi_{i,j}(a)$, which gives (4). \blacksquare

Remark 2.1. Note that states φ and (φ_j) share together the property of extending the array of states $(\varphi_{i,j})$. Thus, φ extends the diagonal states $\varphi_{j,j}$ for all j , but it does not extend the non-diagonal states $\varphi_{i,j}$ for $i \neq j$ since $\pi_M(a)\xi = 0$ for any $a \in \mathcal{A}_{i,j}$. In turn, φ_j extends $\varphi_{i,j}$ for $i \neq j$, but it does not extend $\varphi_{j,j}$ or any of the other diagonal states. This is a natural consequence of differences in the definitions of the diagonal and non-diagonal partial isometries. Moreover, it is clear that for each φ_j there exists $b_j \in \mathcal{A}_{j,j} \cap \text{Ker}\varphi$ such that

$$\varphi_j(w) = \varphi(b_j^* w b_j)$$

for any $w \in \sqcup_{i,j} \mathcal{A}_{i,j}$. Therefore we can reduce computations of mixed moments in the state φ_j to computations of mixed moments in the state φ .

Finally, let us denote by $\lambda(\mathcal{I})$ the unital commutative $*$ -subalgebra of $B(\mathcal{H})$ generated by the $\lambda(1_{i,j})$, where $i, j \in I$. By abuse of notation, $\lambda(1_{i,j})$ will also be denoted by $1_{i,j}$. Observe that these projections are not mutually orthogonal and that is why it is often convenient to use their subprojections $r_{i,j}$ and $s_{i,j}$.

3. STRONGLY MATRICIALLY FREE PRODUCTS

Of special importance is the subspace of the matricially free Fock space, called the ‘strongly matricially free Fock space’, in which the diagonal Hilbert spaces appear only at the end of tensor products. The main reason is that it is related to both free and monotone Fock spaces. We also study the associated product states which can be viewed as direct generalizations of both free and monotone products of states.

Definition 3.1. By the *strongly matricially free Fock space* over $\hat{\mathcal{H}} := (\mathcal{H}_{i,j})$ we understand the subspace of $\mathcal{M}(\hat{\mathcal{H}})$ of the form

$$\mathcal{R}(\hat{\mathcal{H}}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{i_1 \neq \dots \neq i_m \\ n_1, \dots, n_m \in \mathbb{N}}} \mathcal{H}_{i_1, i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2, i_3}^{\otimes n_2} \otimes \dots \otimes \mathcal{H}_{i_m, i_m}^{\otimes n_m}.$$

with the canonical inner product.

A justification for the word ‘strong’ is that in this case the words $i_1 i_2 \dots i_m$ which label the tensor products in the above definition satisfy $i_1 \neq i_2 \neq \dots \neq i_m$.

Example 3.1. The simplest space of this type is associated with a 2-dimensional square array $\hat{\mathcal{H}}$. Then

$$\mathcal{R}(\hat{\mathcal{H}}) = \bigoplus_{m=0}^{\infty} \mathcal{R}^{(m)}(\hat{\mathcal{H}}),$$

where the first few summands are of the form

$$\begin{aligned} \mathcal{R}^{(0)}(\hat{\mathcal{H}}) &= \mathbb{C}\Omega \\ \mathcal{R}^{(1)}(\hat{\mathcal{H}}) &= \mathcal{H}_{1,1} \oplus \mathcal{H}_{2,2} \\ \mathcal{R}^{(2)}(\hat{\mathcal{H}}) &= \mathcal{H}_{1,1}^{\otimes 2} \oplus \mathcal{H}_{2,2}^{\otimes 2} \oplus (\mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}) \oplus (\mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}) \\ \mathcal{R}^{(3)}(\hat{\mathcal{H}}) &= \mathcal{H}_{1,1}^{\otimes 3} \oplus \mathcal{H}_{2,2}^{\otimes 3} \oplus (\mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}^{\otimes 2}) \oplus (\mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}^{\otimes 2}) \oplus (\mathcal{H}_{2,1}^{\otimes 2} \otimes \mathcal{H}_{1,1}) \\ &\quad \oplus (\mathcal{H}_{1,2}^{\otimes 2} \otimes \mathcal{H}_{2,2}) \oplus (\mathcal{H}_{1,2} \otimes \mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}) \oplus (\mathcal{H}_{2,1} \otimes \mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}), \end{aligned}$$

etc. In contrast to $\mathcal{M}(\hat{\mathcal{H}})$, we do not have tensor products like $\mathcal{H}_{2,2} \otimes \mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}$ and $\mathcal{H}_{1,1} \otimes \mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}$ in the summand of the third order.

Remark 3.1. For a given array of Hilbert spaces $\hat{\mathcal{H}} = (\mathcal{H}_{i,j})$, we have inclusions

$$\mathcal{R}(\hat{\mathcal{H}}) \subseteq \mathcal{M}(\hat{\mathcal{H}}) \subseteq \mathcal{F}\left(\bigoplus_{i,j} \mathcal{H}_{i,j}\right)$$

which, in most cases, are proper. Moreover, if we have a square array and $\mathcal{H}_{i,j} \cong \mathcal{H}_i$ for any $i, j \in I$, where $(\mathcal{H}_i)_{i \in I}$ is a family of Hilbert spaces, then there is a natural isomorphism

$$\mathcal{R}(\hat{\mathcal{H}}) \cong \mathcal{F}\left(\bigoplus_{i \in I} \mathcal{H}_i\right)$$

since $\mathcal{H}_{i_1, i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2, i_3}^{\otimes n_2} \otimes \dots \otimes \mathcal{H}_{i_m, i_m}^{\otimes n_m} \cong \mathcal{H}_{i_1}^{\otimes n_1} \otimes \mathcal{H}_{i_2}^{\otimes n_2} \otimes \dots \otimes \mathcal{H}_{i_m}^{\otimes n_m}$ for any $i_1, i_2, \dots, i_m \in I$, $n_1, \dots, n_m, m \in \mathbb{N}$. Similarly, if we have a lower-triangular array and $\mathcal{H}_{i,j} \cong \mathcal{H}_i$ for any $i \geq j$, then $\mathcal{R}(\hat{\mathcal{H}})$ is isomorphic to the monotone Fock space.

Moreover, as expected, there is a product of Hilbert spaces related to the strongly matricially free Fock space, and an analog of Proposition 2.1 holds.

Definition 3.2. By the *strongly matricially free product* of $(\mathcal{H}_{i,j}, \xi_{i,j})$ we understand the pair (\mathcal{G}, ξ) , where \mathcal{G} is the subspace of \mathcal{H} of the form

$$\mathcal{G} = \mathbb{C}\xi \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{i_1 \neq \dots \neq i_m} \mathcal{H}_{i_1, i_2}^0 \otimes \mathcal{H}_{i_2, i_3}^0 \otimes \dots \otimes \mathcal{H}_{i_m, i_m}^0$$

We denote it $(\mathcal{G}, \xi) = *_{i,j}^R(\mathcal{H}_{i,j}, \xi_{i,j})$.

If we consider a family of unital C^* -algebras $(\mathcal{A}_i)_{i \in I}$, each equipped with a family of states $(\varphi_{i,j})_{j \in I}$, then we can look at this product space as follows. If $(\mathcal{H}_{i,j}, \pi_{i,j}, \xi_{i,j})$ is the GNS triple associated with the pair $(\mathcal{A}_i, \varphi_{i,j})$, then the Hilbert space $\mathcal{H}_{i,j}$ (as well as the corresponding state and representation) taken as the representation space for the algebra \mathcal{A}_i at some given tensor site depends on the algebra \mathcal{A}_j represented at the following tensor site on the space $\mathcal{H}_{j,k}$ for some k . It is worth noting that in this framework we can assume that for fixed $i \in I$ all vectors $\xi_{i,j}, j \in I$, are identified since we can take the tensor product of Hilbert spaces $\otimes_j \mathcal{H}_{i,j}$ and set $\xi_i = \otimes_{j \in I} \xi_{i,j}$ for each

$i \in I$. It is not hard to see that in this framework our model is related to freeness with infinitely many states [8,9].

The construction of the product state is similar to that in the usual case. The only difference in all definitions is that the sets Λ_n are replaced by

$$\Gamma_n = \{((i_1, i_2), (i_2, i_3), \dots, (i_n, i_{n+1})) : i_1 \neq i_2 \neq \dots \neq i_{n+1}\}$$

and their union Λ by $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. Note that the conditions which define the Γ_n are stronger than those which define the Λ_n and therefore all objects constructed in the strong case are obtained from the standard ones by a projection-type operation.

The partial isometries in the strong case, denoted by $W_{i,j}$, remind the $V_{i,j}$ except that they refer to \mathcal{G} rather than \mathcal{H} . In particular, the diagonal partial isometries $W_{j,j} : \mathcal{H}_{j,j} \otimes \mathcal{G}(j, j) \rightarrow \mathcal{G}$ are given by

$$\xi_{j,j} \otimes \xi \rightarrow \xi \quad \text{and} \quad \mathcal{H}_{j,j}^0 \otimes \xi \rightarrow \mathcal{H}_{j,j}^0$$

where $\mathcal{G}(j, j) = \mathbb{C}\xi$ for any j , whereas the non-diagonal partial isometries $W_{i,j}$ and the associated subspaces $\mathcal{G}(i, j)$ are similar to the $\mathcal{H}(i, j)$.

Definition 3.3. Let $\rho_{i,j}$ be the $*$ -representation of $\mathcal{A}_{i,j}$ defined as in the standard case, with $V_{i,j}$ replaced by $W_{i,j}$ for any i, j . The corresponding *strongly matricially free product representation* $\pi_R = *_{i,j}^R \pi_{i,j}$ and *strongly matricially free product of states* $*_{i,j}^R \varphi_{i,j}$ are defined in terms of the $\rho_{i,j}$ as in the standard case.

The basic results on the strongly matricially free product of states are similar to those in Propositions 2.2-2.4 and therefore we state them in an abbreviated form without a proof.

Proposition 3.1. *Let φ be the strongly matricially free product of states $(\varphi_{i,j})$ and let φ_j be the state associated with any unit vector from $\mathcal{H}_{j,j}^0$ which is in the range of $\rho_{j,j}(\mathcal{A}_{j,j})$, $j \in I$. Then the statements of Propositions 2.2-2.3 remain true, with Λ replaced by Γ .*

As we mentioned earlier, one of the advantages of using the strong structures is that they are straightforward generalizations of those in free probability (in the case of square arrays) and monotone probability (in the case of triangular arrays).

Remark 3.2. Using the strongly matricially free product of states, we can obtain the free product of states if we take a square array. Namely, if $\mathcal{A}_{i,j} = \mathcal{A}_i$ and $\varphi_{i,j} = \varphi_i$ for all $i, j \in I$, then, on the level of Hilbert spaces, we have $\mathcal{H}_{i,j} = \mathcal{H}_i$ and $\xi_{i,j} = \xi_i$ for all $i, j \in I$. Thus

$$(\mathcal{G}, \xi) \cong *_{i \in I} (\mathcal{H}_i, \xi_i)$$

and $\rho_i(a) := \sum_j \rho_{i,j}(a)$, $a \in \mathcal{A}_i$, understood as the strong limit, extends to a (unital) $*$ -representation of \mathcal{A}_i on \mathcal{H} . Then $\rho = \sqcup_{i \in I} \rho_i$ agrees with the free product representation of $\sqcup_{i \in I} \mathcal{A}_i$ on \mathcal{G} and therefore the corresponding strongly matricially free product of states gives the free product of states. If $\varphi_{i,i} = \varphi_i$ and $\varphi_{i,j} = \psi_i$ for any $i \neq j$, we obtain the conditionally free product of states.

Remark 3.3. The monotone product of states [17] is obtained from the strongly matricially free product of states when we take a triangular array T , with $\mathcal{A}_{i,j} = \mathcal{A}_i$ and

$\varphi_{i,j} = \varphi_i$ for all $(i, j) \in T \subset I \times I$ (it can be obtained from the square array by setting $\mathcal{A}_{i,j} = 0$ for $i < j$), where I is a totally ordered set. Then $\mathcal{H}_{i,j} = \mathcal{H}_i$ and $\xi_{i,j} = \xi_i$ for $i \geq j$ and

$$(\mathcal{G}, \xi) \cong_{\triangleright_{i \in I}} (\mathcal{H}_i, \xi_i),$$

the expression on the right-hand side being the monotone product of Hilbert spaces. Moreover, $\tau_i(a) := \sum_{j \leq i} \rho_{i,j}(a)$ extends to a (non-unital) $*$ -representation of \mathcal{A}_i on \mathcal{G} , such that $\tau = \sqcup_{i \in I} \tau_i$ agrees with the monotone product representation of $\sqcup_{i \in I} \mathcal{A}_i$ on \mathcal{G} and therefore the corresponding strongly matricially free product of states gives the monotone product of states.

4. MATRICIAL FREENESS

Guided by the notion of the matricially free product of states, we shall introduce now the associated concept of independence called ‘matricial freeness’, and closely related to it, ‘strong matricial freeness’. They involve arrays of noncommutative probability spaces and to some extent they remind models with many states [6,8,9], but they cannot be reduced in a natural way to any of these (freeness with infinitely many states has some non-empty intersection with ‘strong matricial freeness’ and conditional freeness is its special case). Moreover, we will study discrete matricially free Fock spaces, both standard and strong.

Let \mathcal{A} be a unital algebra with an array $(\mathcal{A}_{i,j})$ of not necessarily unital subalgebras of \mathcal{A} . Assume that each $\mathcal{A}_{i,j}$ has an internal unit $1_{i,j}$ and that the subalgebra \mathcal{I} of \mathcal{A} generated by all internal units is commutative. Further, let φ be a distinguished state on \mathcal{A} such that $\varphi(1_{i,j}) = \delta_{i,j}$ for any i, j and let $\{\varphi_j : j \in I\}$ be a family of additional states on \mathcal{A} such that $\varphi_j(1_{i,k}) = \delta_{j,k}$ for any i, j, k , which will be called *conditions*. Here, by a state on \mathcal{A} we understand a normalized linear functional (if \mathcal{A} is a $*$ -algebra, we require this functional to be positive).

In this situation it is convenient to form an array $(\varphi_{i,j})$ of states on \mathcal{A} by

$$\varphi_{j,j} = \varphi \quad \text{and} \quad \varphi_{i,j} = \varphi_j \quad \text{for } i \neq j$$

which will be said to be *defined* by the state φ and the conditions φ_j .

Definition 4.1. Let $(\varphi_{i,j})$ be defined by the state φ and the conditions φ_j . We say that $(1_{i,j})$ is a *matricially free array of units* associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$ if

- (1) $\varphi(u_1 a u_2) = \varphi(u_1) \varphi(a) \varphi(u_2)$ for any $a \in \mathcal{A}$ and $u_1, u_2 \in \mathcal{I}$,
- (2) if $a_k \in \mathcal{A}_{i_k, j_k} \cap \text{Ker} \varphi_{i_k, j_k}$, where $r < k \leq n$ and $r < n$, then

$$\varphi(a 1_{i_r, j_r} a_{r+1} \dots a_n) = \begin{cases} \varphi(a a_{r+1} \dots a_n) & \text{if } ((i_r, j_r), \dots, (i_n, j_n)) \in \Lambda \\ 0 & \text{otherwise} \end{cases}.$$

where $a \in \mathcal{A}$ is arbitrary and $(i_r, j_r) \neq \dots \neq (i_n, j_n)$.

The array $(1_{i,j})$ is called a *strongly matricially free array of units* if Λ is replaced by Γ .

The above definition enables us to define the concepts of *matricial freeness* and its strong version called *strong matricial freeness*. They both bear some resemblance to freeness, but the main difference is that the identified unit in the context of freeness is replaced by the (strongly) matricially free array of units. In fact, we will see later

that it is the strong matricial freeness which can be viewed as a direct generalization of freeness.

Definition 4.2. We say that $(\mathcal{A}_{i,j})$ is *matricially free* with respect to $(\varphi_{i,j})$ if

- (1) for any $a_k \in \text{Ker} \varphi_{i_k, j_k} \cap \mathcal{A}_{i_k, j_k}$, where $k \in [n]$ and $(i_1, j_1) \neq \dots \neq (i_n, j_n)$,
- $$\varphi(a_1 a_2 \dots a_n) = 0$$

- (2) $(1_{i,j})$ is a matricially free array of units associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$.

In an analogous manner we define *strongly matricially free* arrays of subalgebras.

Definition 4.3. The array of variables $(a_{i,j})$ in a unital algebra \mathcal{A} will be called *(strongly) matricially free* with respect to the array $(\varphi_{i,j})$ defined by a state φ and the conditions φ_j if there exists an array of (strongly) matricially free array of units $(1_{i,j})$ in \mathcal{A} such that the array of algebras $(\mathbb{C}[a_{i,j}, 1_{i,j}])$ is (strongly) matricially free with respect to $(\varphi_{i,j})$.

Using the above definitions, we can compute mixed moments of arbitrary matricially free random variables. Namely, writing $a_k = a_k^0 + \varphi_{i_k, j_k}(a_k)1_{i_k, j_k}$ for each $k \in [n]$, we obtain a recursion

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \sum_{1 \leq k \leq n} \varphi_{i_k, j_k}(a_k) \varphi(a_1^0 \dots 1_{i_k, j_k} \dots a_n^0) \\ &+ \sum_{1 \leq k < l \leq n} \varphi_{i_k, j_k}(a_k) \varphi_{i_l, j_l}(a_l) \varphi(a_1^0 \dots 1_{i_k, j_k} \dots 1_{i_l, j_l} \dots a_n^0) \\ &+ \dots \\ &+ \varphi_{i_1, j_1}(a_1) \dots \varphi_{i_n, j_n}(a_n) \varphi(1_{i_1, j_1} \dots 1_{i_n, j_n}) \end{aligned}$$

for any $(i_1, j_1) \neq \dots \neq (i_n, j_n)$. It is easy to see that the mixed moments on the right-hand side, written here in a slightly simplified manner, reduce to moments of orders smaller than n . Note that these depend on $\varphi|_{\mathcal{I}}$ in an essential way. That is why the states φ and the φ_j of Section 2 give different mixed moments, although they have similar properties.

Remark 4.1. If \mathcal{A} is a unital $*$ -algebra, then, in addition, we require that the functionals $\varphi_{i,j}$ are positive, the $\mathcal{A}_{i,j}$ are $*$ -subalgebras and the $1_{i,j}$ are projections. Then an array of variables $(a_{i,j})$ will be called $*$ -(strongly) *matricially free* if the array of $*$ -algebras $(\mathbb{C}\langle a_{i,j}, a_{i,j}^*, 1_{i,j} \rangle)$ is (strongly) matricially free.

Example 4.1. Let us assume that we have a two-dimensional square array of variables, namely $a_{1,1} = a, a_{1,2} = a', a_{2,1} = b', a_{2,2} = b$, which are strongly matricially free with respect to the array $(\varphi_{i,j})$ defined by φ and the pair (φ_1, φ_2) . Using the above recursion and then Definitions 4.1-4.2, we obtain

$$\begin{aligned} \varphi(ab'ab) &= \varphi(b)\varphi_1(b')(\varphi(a^2) - \varphi^2(a)), \\ \varphi(abab) &= \varphi^2(a)\varphi^2(b), \\ \varphi(aba'b) &= \varphi(a)\varphi_2(a')(\varphi(b^2) - \varphi^2(b)). \end{aligned}$$

Note that if we assume that $\varphi_1 = \varphi_2 = \psi$, then the sum of these moments gives $\varphi(abab)$ for a, b conditionally independent with respect to (φ, ψ) . In fact, the above sum is equal

to $\varphi((a + a')(b + b')(a + a')(b + b'))$. In turn, if we take the lower-triangular subarray and take φ_1 and φ -distributions to be equal, the sum of the first two moments (the third one is zero since there is no a') gives $\varphi(abab)$ for a, b monotone independent with respect to φ . Similar agreements hold for mixed moments of higher orders, which is in agreement with Remarks 3.2-3.3.

Definition 4.4. By a *discrete matricially free Fock space* we understand $\mathcal{M} = \mathcal{M}(\hat{\mathcal{H}})$, where $\mathcal{H}_{i,j} = \mathbb{C}e_{i,j}$ for any $(i, j) \in \mathbb{N}$, and the array $(e_{i,j})$ forms an orthonormal basis of some Hilbert space. In an analogous manner we define the *discrete strongly matricially free Fock space* $\mathcal{R} = \mathcal{R}(\hat{\mathcal{H}})$.

Both \mathcal{M} and \mathcal{R} are subspaces of the discrete free Fock space $\mathcal{F}(\bigoplus_{i,j} \mathbb{C}e_{i,j})$ and thus allow for the canonical action of free creation and annihilation operators. In order to specify the arrays of units, let us distinguish two types of their subspaces.

In the case of \mathcal{M} these are

- (1) $\mathcal{M}(i, j)$, spanned by simple tensors which begin with $e_{j,k}$ for some k , where $(j, k) \neq (i, j)$, and, in addition, by Ω if $i = j$,
- (2) $\mathcal{K}(i, j)$, spanned by vectors which begin with $e_{i,j}$, where i, j are arbitrary,

and the direct sum $\mathcal{K}(i, j) \oplus \mathcal{M}(i, j)$ is the subspace of \mathcal{M} onto which the $*$ -algebra generated by $\ell(e_{i,j})$ restricted to \mathcal{M} acts non-trivially, where $\ell(e_{i,j})$ denotes the canonical free creation operator associated with vector $e_{i,j}$.

The canonical projections onto such direct sums are natural candidates for the matricially free units $1_{i,j}$ and therefore we set

$$1_{i,j} := P_{\mathcal{K}(i,j) \oplus \mathcal{M}(i,j)} \quad \text{and} \quad \ell_{i,j} = \ell(e_{i,j})1_{i,j}$$

for any i, j . The adjoint of $\ell_{i,j}$ will be denoted by $\ell_{i,j}^*$. Note that the projection $1_{i,j}$ is an internal unit in the $*$ -algebra $\mathcal{A}_{i,j} = \mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^* \rangle$ and $\ell_{i,j}^* \ell_{i,j} = 1_{i,j}$ for any i, j . However, in the algebra $\mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^* : i, j \in \mathbb{N} \rangle$ there are more relations as the proposition given below demonstrates.

Proposition 4.1. *In the algebra $\mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^* : i, j \in \mathbb{N} \rangle$ the following relations hold:*

- (1) $\ell_{i,j}^* \ell_{i,j} = 1_{i,j}$ for any i, j ,
- (2) $\ell_{i,j}^* \ell_{k,l} = 0$, $\ell_{i,j} \ell_{k,l} = 0$ and $1_{i,j} \ell_{k,l} = 0$ whenever $(i, j) \neq (k, l)$ and $j \neq k$,
- (3) $1_{i,j} \ell_{j,k} = \ell_{j,k}$ for any j, k .

Proof. We omit the elementary proof. ■

In the case of \mathcal{R} , we proceed in a completely analogous fashion and distinguish the following subspaces:

$$\mathcal{R}(i, j) = \mathcal{M}(i, j) \cap \mathcal{R} \quad \text{and} \quad \mathcal{L}(i, j) = \mathcal{K}(i, j) \cap \mathcal{R}$$

which lead to the definitions of the strongly matricially free array of units and of the creation operators, respectively,

$$1_{i,j} = P_{\mathcal{L}(i,j) \oplus \mathcal{R}(i,j)} \quad \text{and} \quad \ell_{i,j} = \ell(e_{i,j})1_{i,j}$$

where, slightly abusing notation, we use the same symbols for the units as before.

Finally, we specify the states: φ will denote the vacuum state associated with Ω , the φ_j will be the states associated with the $e_{j,j}$, $j \in \mathbb{N}$, and $(\varphi_{i,j})$ will be the array defined by φ and the φ_j . Here, the same notation is used for states on $B(\mathcal{M})$ and $B(\mathcal{R})$.

Proposition 4.2. *The array of *-subalgebras $\mathcal{A}_{i,j} = \mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^* \rangle$ of $B(\mathcal{M})$, where $i, j \in \mathbb{N}$, is matricially free with respect to $(\varphi_{i,j})$. The array of *-subalgebras $\mathcal{B}_{i,j} = \mathbb{C}\langle k_{i,j}, k_{i,j}^* \rangle$ of $B(\mathcal{R})$, where $i, j \in \mathbb{N}$, is strongly matricially free with respect to $(\varphi_{i,j})$.*

Proof. The proof is similar to that of Voiculescu for the discrete free Fock space given in [21]. We shall look at the case of $(\mathcal{A}_{i,j})$ since the case of $(\mathcal{B}_{i,j})$ is analogous. Each algebra $\mathcal{A}_{i,j}$ is spanned by operators of the form

$$\ell_{i,j}^q \ell_{i,j}^{*p}, \text{ where } p + q > 0,$$

and the projection $1_{i,j}$. However, the corresponding moments vanish:

$$\varphi_{i,j}(\ell_{i,j}^q \ell_{i,j}^{*p}) = 0$$

for any i, j since $p + q > 0$. Moreover, $\varphi_{i,j}(1_{i,j}) = 1$ for any i, j . Therefore, in order to show that condition (1) of Definition 4.2 holds, it is enough to show that

$$\varphi(\ell_{i_1,j_1}^{q_1} \ell_{i_1,j_1}^{*p_1} \dots \ell_{i_n,j_n}^{q_n} \ell_{i_n,j_n}^{*p_n}) = 0$$

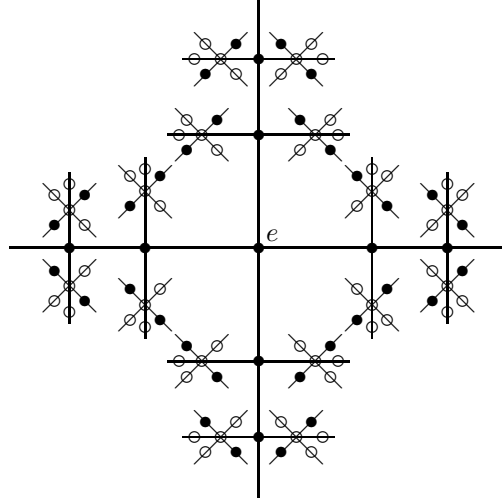
whenever $(i_1, j_1) \neq \dots \neq (i_n, j_n)$ and $p_1 + q_1 > 0, \dots, p_n + q_n > 0$. The same argument as in [21] allows us to reduce the proof to the case when $q_1 = \dots = q_n = 0$, which implies that $p_1 > 0, \dots, p_n > 0$. But then the moment clearly vanishes. This proves condition (1) of Definition 4.2. Condition (2) follows easily from the definition of the projections $1_{i,j}$ in view of the relations given in Proposition 4.1. This completes the proof. \blacksquare

Example 4.2. For $i, j \in I$, let $G_{i,j} \cong F(1)$ be the free group on one generator $g_{i,j}$ with unit $\epsilon_{i,j}$, and let $\lambda_{i,j}$ be the corresponding unitary operator on the space $l^2(G_{i,j})$ given by $\lambda_{i,j}(g)\delta(h) = \delta(gh)$, where $\{\delta(g) : g \in G_{i,j}\}$ is the canonical basis. Consider the subspace l_M^2 of $l^2(*_{i,j}G_{i,j})$ spanned by vectors of the form $\delta(g)$, where g is either the unit e of the free product $*_{i,j}G_{i,j}$, or a product of the form $g_1 g_2 \dots g_m$, where $g_k \in G_{i_k, i_{k+1}}^0 := G_{i_k, i_{k+1}} \setminus \{\epsilon_{i_k, i_{k+1}}\}$ for each k , with $(i_1, i_2) \neq \dots \neq (i_m, i_{m+1})$ and $i_{m+1} = i_m$. The space l_M^2 can be viewed as the space of square integrable functions on the (non-existent) ‘matricially free product of groups’. Let $1_{i,j}$ denote the projection from $l^2(*_{i,j}G_{i,j})$ onto the subspace of l_M^2 spanned by vectors $\delta(g)$, where g begins with an element from $G_{i,j}^0$ or $G_{j,k}^0$ for some k or, in the case of $i = j$, also $g = e$. Define

$$\tilde{\lambda}_{i,j}(g) = \lambda_{i,j}(g)1_{i,j}$$

for $g \in G_{i,j}$. Then the array $(\mathcal{A}_{i,j})$, where $\mathcal{A}_{i,j}$ is the *-subalgebra of $B(l^2(*_{i,j}G_{i,j}))$ generated by $\tilde{\lambda}_{i,j}(g_{i,j})$ and $1_{i,j}$, with the standard involution, is matricially free with respect to the array $(\varphi_{i,j})$, where the diagonal states coincide with $\varphi(\cdot) = \langle \delta(e), \delta(e) \rangle$ and the non-diagonal ones in the j -th column coincide with $\varphi_j(\cdot) = \langle \delta(g_{j,j}), \delta(g_{j,j}) \rangle$.

Example 4.3. In the above example take an n -dimensional square array of copies of $F(1)$ and denote their generators by $g_{i,j}$, where $i, j \in [n]$. For each natural n we form a tree (a subtree of the homogenous tree \mathbb{H}_{2n^2}) which corresponds to the ‘matricially

FIGURE 1. Matricially free analog of \mathbb{H}_4

free product of n^2 free groups'. Suppose the root e corresponds to the 'father'. We distinguish 'sons' and 'daughters' in each 'generation' which correspond to the left action of $g_{j,j}$ or $g_{j,j}^{-1}$, and $g_{i,j}$ or $g_{i,j}^{-1}$, respectively, where $i \neq j$. The rules of drawing the tree follow from matricial freeness and are the following: each 'son' has 1 'son' and $2n-2$ 'daughters', whereas each 'daughter' has 2 'sons' and $2n-1$ 'daughters'. Therefore, 'sons' and 'daughters' correspond to vertices of valencies $2n$ and $2n+2$, respectively. In Figure 1 we draw such a tree for $n=2$ (black and empty circles are assigned to 'sons' and 'daughters', respectively). If we make an additional assumption, for instance that 'daughters' cannot have 'sons' (this fact corresponds to strong matricial freeness, where diagonal generators kill words beginning with the non-diagonal ones), we recover \mathbb{H}_{2n} of free probability (cf. Remark 3.1).

Example 4.4. Let $\mathcal{R}(\hat{\mathcal{H}})$ be the discrete strongly matricially free Fock space and let $\mathcal{R}(\hat{\mathcal{H}}) \cong \mathcal{F}(\bigoplus_j \mathbb{C}e_j)$ be the natural isomorphism of Remark 3.1, where $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis of some Hilbert space. If $(w_{i,j})$ is an infinite matrix with non-negative parameters p and q above and below the main diagonal, respectively, and 1's on the diagonal, then this it is easy to see that the (p,q) -creation operators studied in [15] can be identified (with the use of this isomorphism) with the strongly convergent sums

$$A_i = \sum_j w_{i,j} k_{i,j}$$

where $i \in \mathbb{N}$ and the (p,q) -annihilation operators are their adjoints. A similar approach can be applied to square arrays of arbitrary Hilbert spaces and $*$ -representations, which leads to some notion of ' (p,q) -independence'. Moreover, it can be carried out for more general matrices $(w_{i,j})$ within the framework of the strong matricial freeness, which generalizes notions of independence of this type.

5. TRACES

In this Section we introduce some real-valued functions on the set of non-crossing pair partitions. These functions are obtained by computing traces of a square real-valued matrix. We will assume later that this matrix has non-negative entries which represent variances of probability measures on the real line $\mathcal{M}_{\mathbb{R}}$ and we will demonstrate that the functions introduced in this Section describe the asymptotics of matricially free random variables in central limit theorems.

Let \mathcal{NC}_m denote the set of non-crossing partitions of the set $[m]$, i.e. if $\pi = \{\pi_1, \pi_2, \dots, \pi_k\} \in \mathcal{NC}_m$, then there are no numbers $i < p < j < q$ such that $i, j \in \pi_r$ and $p, q \in \pi_s$ for $r \neq s$. The block π_r is *inner* with respect to π_s if $p < i < q$ for any $i \in \pi_r$ and $p, q \in \pi_s$ (then π_s is *outer* with respect to π_r). It is clear that if π_r has outer blocks, then there exists a unique block among them, say π_s , which is *nearest* to π_r , i.e. if another block, say π_t , is outer with respect to π_r , then we must have $a < p < b$ for any $a, b \in \pi_t$ and $p \in \pi_s$. Then the pair (π_r, π_s) is called the *nearest inner-outer pair* of blocks.

Let \mathcal{NCC}_m denote the set of *non-crossing covered pair partitions* of $[m]$, by which we understand the subset of \mathcal{NC}_m consisting of those partitions in which 1 and m belong to the same block (if $m = 1$, we understand that the partition consists of one block). In terms of diagrams, all blocks of $\pi \in \mathcal{NCC}_m$, where $m > 1$, are covered by the block containing 1 and m . We denote by \mathcal{NC}_m^2 and \mathcal{NCC}_m^2 the sets of non-crossing pair partitions of $[m]$ and non-crossing covered pair-partitions of $[m]$, respectively, and we set $\mathcal{NC}^2 = \bigcup_{m=1}^{\infty} \mathcal{NC}_m^2$ and $\mathcal{NCC}^2 = \bigcup_{m=1}^{\infty} \mathcal{NCC}_m^2$.

It is easy to see that each $\pi \in \mathcal{NC}_m$ can be decomposed as

$$(5.1) \quad \pi = \pi^{(1)} \cup \pi^{(2)} \cup \dots \cup \pi^{(p)}$$

where $\pi^{(1)}, \dots, \pi^{(p)}$ are non-crossing covered partitions of subintervals I_1, I_2, \dots, I_p of $[m]$ whose union gives $[m]$. By a partition of a set I consisting of r elements we understand the corresponding partition of $[r]$.

On the other hand, each $\pi \in \mathcal{NCC}_m$ can be decomposed as

$$(5.2) \quad \pi = \pi^{(0)} \cup \pi^{(1)} \cup \dots \cup \pi^{(r)}$$

where $\pi^{(0)}$ is the block containing 1 and m and $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(r)}$ are non-crossing covered partitions of subintervals I_1, I_2, \dots, I_r of the set $\{2, \dots, m-1\}$.

Consider now a square real-valued matrix $V = (v_{i,j}) \in M_n(\mathbb{R})$, where $n \in \mathbb{N} \cup \{\infty\}$. The usual trace and the normalized trace will be denoted

$$\mathrm{Tr}(V) = \sum_{j=1}^n v_{j,j} \quad \text{and} \quad \mathrm{tr}(V) = \frac{1}{n} \sum_{j=1}^n v_{j,j},$$

respectively. For finite n , we define the ‘diagonalization mapping’

$$\tau : M_n(\mathbb{R}) \rightarrow D_n(\mathbb{R}), \quad \tau(V) = \mathrm{diag}\left(\sum_j v_{j,1}, \dots, \sum_j v_{j,n}\right)$$

where $D_n(\mathbb{R})$ is the set of square diagonal real-valued matrices of dimension n (by abuse of notation, the same symbol τ is used for all n). In other words, τ computes the sum of all elements of V in each column separately and puts this value on the diagonal.

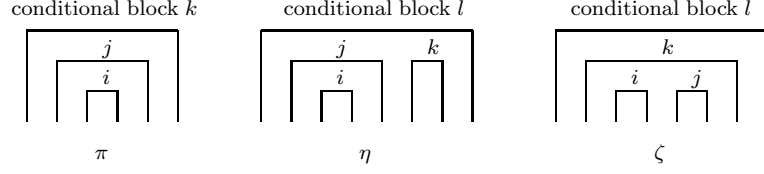


FIGURE 2. Colored partitions with conditional blocks.

Using the above trace operations, we shall define two real-valued functions on the set \mathcal{NC}^2 , denoted v and v_0 , associated with given $V \in M_n(\mathbb{R})$. Although there is a close similarity between these functions when restricted to \mathcal{NCC}^2 (in particular, they are defined as traces of certain matrix-valued quasi-multiplicative functions), note that they are extended to \mathcal{NC}^2 in two different ways.

Definition 5.1. For a given matrix $V \in M_n(\mathbb{R})$, we define a mapping from \mathcal{NC}^2 to $D_n(\mathbb{R})$ by assigning to each $\pi \in \mathcal{NC}^2$ the matrix $V(\pi)$ by the following recursion:

- (1) if π consists of one block, we set $V(\pi) = \tau(V)$,
- (2) if $\pi \in \mathcal{NCC}^2$ consists of more than one block, then

$$(5.3) \quad V(\pi) = \tau(V(\pi^{(1)}) \dots V(\pi^{(r)})V)$$

according to the decomposition (5.2),

- (3) if $\pi \in \mathcal{NC}^2$, then

$$(5.4) \quad V(\pi) = V(\pi^{(1)})V(\pi^{(2)}) \dots V(\pi^{(p)})$$

according to the decomposition (5.1).

Let $v : \mathcal{NC}^2 \rightarrow \mathbb{R}$ be the function defined by $v(\pi) = \text{tr}(V(\pi))$.

Example 5.1. For some $V \in M_n(\mathbb{R})$, consider three partitions given in Fig.2. Color each block by a number from the set $[n]$. Computation of the corresponding values of the function v gives

$$\begin{aligned} v(\pi) &= \text{tr}(\tau(\tau(V)V)) = \frac{1}{n} \sum_{i,j,k} v_{i,j} v_{j,k} \\ v(\eta) &= \text{tr}(\tau(\tau(V)V) \tau(V)) = \frac{1}{n} \sum_{i,j,k,l} v_{i,j} v_{j,k} v_{k,l} \\ v(\zeta) &= \text{tr}(\tau(\tau(V)\tau(V)V)) = \frac{1}{n} \sum_{i,j,k,l} v_{i,k} v_{j,k} v_{k,l} \end{aligned}$$

One can see that to each nearest inner-outer pair of blocks (π_r, π_s) we assign the matrix element $v_{p,q}$, where p and q are the colors of π_r and π_s respectively. Moreover, if a block does not have any outer blocks and is colored by q , then we assign to it the matrix element $v_{q,t}$, where t is assumed to be the same for all such blocks (one can imagine that we have an additional ‘conditional block’ colored by t which covers all other blocks). At the end we sum over all colorings.

The function v_0 is defined on the set \mathcal{NCC}^2 in a very similar manner, except that we replace the right multiplier V in (5.3) by its main diagonal

$$V_0 := \text{diag}(v_{1,1}, \dots, v_{n,n})$$

which corresponds to changing only the contribution of the covering block. Then we extend v_0 to all of \mathcal{NCC}^2 by multiplicativity of v_0 .

Definition 5.2. For a given matrix $V \in M_n(\mathbb{R})$, we define a mapping from \mathcal{NCC}^2 to $D_n(\mathbb{R})$ by assigning to each $\pi \in \mathcal{NCC}^2$ the matrix $V_0(\pi)$ by the following recursion:

- (1) if π consists of one block, we set $V_0(\pi) = V_0$
- (2) if $\pi \in \mathcal{NCC}^2$ consists of more than one block, then

$$(5.5) \quad V_0(\pi) = \tau(V(\pi^{(1)})) \dots V(\pi^{(r)})V_0$$

according to the decomposition (5.2).

Let $v_0 : \mathcal{NCC}^2 \rightarrow \mathbb{R}$ be the function defined by $v_0(\pi) = \text{Tr}(V_0(\pi))$ for $\pi \in \mathcal{NCC}^2$, extended to \mathcal{NCC}^2 by multiplicativity $v_0(\pi) = v_0(\pi^{(1)}) \dots v_0(\pi^{(p)})$ according to the decomposition (5.1).

Example 5.2. Let us compute the values of v_0 corresponding to the partitions in Fig.2. We have

$$\begin{aligned} v_0(\pi) &= \text{Tr}(\tau(V)V_0) = \sum_{i,j} v_{i,j}v_{j,j} \\ v_0(\eta) &= \text{Tr}(\tau(V)V_0)\text{Tr}(V_0) = \sum_{i,j,k} v_{i,j}v_{j,j}v_{k,k} \\ v_0(\zeta) &= \text{Tr}(\tau(\tau(V)\tau(V)V_0)) = \sum_{i,j,k} v_{i,k}v_{j,k}v_{k,k} \end{aligned}$$

Note that the main difference (apart from normalization) between $v_0(\pi)$ and $v(\pi)$ concerns the blocks which do not have outer blocks. Here, if such a block is colored by q , we assign to it the matrix element $v_{q,q}$ and we do not use ‘conditional blocks’.

Below we shall prove a lemma which gives explicit combinatorial formulas for $v(\pi)$ and $v_0(\pi)$. If $\pi = \{\pi_1, \dots, \pi_k\} \in \mathcal{NCC}_{2k}^2$, we color the blocks of π and the conditional block π_0 by the set $[n]$. The *coloring* of the partition $\pi \cup \{\pi_0\}$ can then be identified with the mapping $f : [k] \cup \{0\} \rightarrow [n]$, where $f(i)$ is interpreted as the color of π_i . The pair (π, f) can then be interpreted as the colored partition associated with π and the mapping f . The set of all colorings of π by the set $[n]$ will be denoted $F_n(\pi)$.

For a given matrix $V = (v_{i,j}) \in M_n(\mathbb{R})$ and given colored partition (π, f) , where π is a non-crossing pair partition consisting of k blocks and f is the coloring of π , we can now define (slightly abusing notation) a natural mapping

$$v_0 : \{(\pi_1, f), \dots, (\pi_k, f)\} \rightarrow \mathbb{R},$$

by the following rule:

$$v_0(\pi_p, f) = v_{i,j} \text{ if } f(p) = i \text{ and } f(\sigma(p)) = j$$

where $\sigma(p)$ is the index of the nearest outer block of π_p if π_p has an outer block, and otherwise $\sigma(p) = p$. In other words, to each block we assign the matrix element whose

first index is the color of that block, and the second index – the color of its nearest outer block.

The multiplicative extension of this function to the class of non-crossing colored partitions gives

$$v_0(\pi, f) = v_0(\pi_1, f)v_0(\pi_2, f) \dots v_0(\pi_k, f).$$

In a similar way we define functions

$$v_j(\pi, f) = v_j(\pi_1, f)v_j(\pi_2, f) \dots v_j(\pi_k, f)$$

for any $j \in [n]$, where $v_j(\pi_p, f) = v_{i,j}$ if $f(p) = i$ and $f(\sigma_j(p)) = j$ where $\sigma_j(p)$ is the index of the nearest outer block of π_p if π_p has an outer block, and otherwise we put $\sigma_j(p) = j$. In the latter case, j is interpreted as the color of the conditional block.

Lemma 5.1. *For any $V \in M_n(\mathbb{R})$, where $n \in \mathbb{N}$, and $\pi \in \mathcal{NC}^2$, it holds that*

$$v_0(\pi) = \sum_{f \in F_n(\pi)} v_0(\pi, f) \quad \text{and} \quad v(\pi) = \frac{1}{n} \sum_{f \in F_n(\pi)} \sum_{j \in [n]} v_j(\pi, f)$$

where the summation over j corresponds to all colorings of the conditional block.

Proof. We provide an induction proof for the function v (the proof for v_0 is similar). The main induction step will be carried out on the level of the (diagonal) matrices $V(\pi)$. We claim that its diagonal entries are of the form

$$(V(\pi))_{q,q} = \sum_{f \in F_n(\pi)} v_q(\pi, f)$$

for any $\pi \in \mathcal{NCC}^2$ and $q \in [n]$. In view of (5.4), the required formula for $v(\pi)$ is then a straightforward consequence of the claim. Of course, if π consists of one block, then

$$(V(\pi))_{q,q} = \sum_j v_{j,q}$$

and thus our assertion easily follows. Assume now that π has $k > 2$ blocks and suppose the assertion holds for non-crossing covered partitions which have less than k blocks. Since π has a decomposition of type (5.2), the assertion holds for $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(r)}$ in this decomposition. We know that the matrix assigned to π has the form (5.3) and therefore the product of (diagonal) matrices corresponding to these subpartitions has (diagonal) matrix elements of the form

$$(V(\pi))_{q,q} = (\tau(WV))_{q,q} = \sum_j W_{j,j} v_{j,q}$$

where q is the color of the conditional block of π and

$$W_{j,j} = \sum_{f_1 \in F_n(\pi^{(1)})} v_j(\pi^{(1)}, f_1) \dots \sum_{f_r \in F_n(\pi^{(r)})} v_j(\pi^{(r)}, f_r)$$

by the inductive assumption. Now, since the blocks of $\pi^{(1)}, \dots, \pi^{(r)}$ are colored independently, we have

$$v_j(\pi^{(1)}, f_1) \dots v_j(\pi^{(r)}, f_r) v_{j,q} = v_q(\pi, f)$$

for a uniquely determined coloring f of the blocks of π in which j can be interpreted as the color of $\pi^{(0)}$ since $\pi^{(0)}$ covers $\pi^{(1)}, \dots, \pi^{(r)}$ and q can be viewed as the color of the conditional block of π . This and the above formula for $W_{j,j}$ gives the desired formula

$$V(\pi)_{q,q} = \sum_{f \in F_n(\pi)} v_q(\pi, f),$$

which proves our claim and thus completes the proof of the theorem. \blacksquare

Under suitable assumptions on V , there is a simple connection between functions v and v_0 if $V_0 = aI_n/n$, where I_n is a unit matrix and a is a positive number. We shall express this relation in terms of the corresponding formal Laurent series, which turn out to be Cauchy transforms of (compactly supported) probability measures on the real line associated with appropriately constructed random variables,

Proposition 5.1. *Let $V \in M_n(\mathbb{R})$ be such that $V_0 = aI_n/n$, where $a > 0$. If $G(z) = \sum_{k=0}^{\infty} a_{2k} z^{-2k-2}$ and $G_0(z) = \sum_{k=0}^{\infty} b_{2k} z^{-2k-2}$ are formal Laurent series, where*

$$a_{2k} = \sum_{\pi \in \mathcal{NC}_{2k}^2} v(\pi) \quad \text{and} \quad b_{2k} = \sum_{\pi \in \mathcal{NC}_{2k}^2} v_0(\pi)$$

and both $v(\pi)$ and $v_0(\pi)$ are associated with V , then $G_0(z) = 1/(z - aG(z))$.

Proof. Observe that in the case when $v_{j,j} = a/n$ for all $j \in [n]$, we have

$$v(\pi) = \frac{v_0(\pi')}{a}$$

for any $\pi \in \mathcal{NC}_m^2$, where the partition $\pi' \in \mathcal{NCC}_{m+2}^2$ is obtained from π by adding to π the block that covers all blocks of π , say $\{0, m+1\}$. This leads to

$$\begin{aligned} A(z) : &= \sum_{m=0}^{\infty} a_{2m} z^{2m} = 1 + \sum_{m=1}^{\infty} \sum_{\pi \in \mathcal{NC}_{2m}^2} v(\pi) z^{2m} \\ &= 1 + \frac{1}{a} \sum_{m=1}^{\infty} \sum_{\pi \in \mathcal{NCC}_{2m+2}^2} v_0(\pi) z^{2m} = \frac{C(z) - 1}{az^2} \end{aligned}$$

where $C(z) = \sum_{m=0}^{\infty} c_{2m} z^{2m}$ and $c_{2m} = \sum_{\pi \in \mathcal{NCC}_{2m}^2} v_0(\pi)$. Now, using the multiplicativity of v_0 , we obtain

$$B(z) := \sum_{m=0}^{\infty} b_{2m} z^{2m} = 1 + \sum_{m=1}^{\infty} (C(z) - 1)^m = \frac{1}{2 - C(z)}$$

which leads to

$$A(z) = \frac{1}{az^2} \left(1 - \frac{1}{B(z)} \right) \quad \text{and} \quad G_0(z) = \frac{1}{z - aG(z)}$$

where

$$G(z) = \frac{1}{z} A\left(\frac{1}{z}\right) \quad \text{and} \quad G_0(z) = \frac{1}{z} B\left(\frac{1}{z}\right)$$

which completes the proof. \blacksquare

6. RANDOM PSEUDOMATRICES

In this Section we will study the asymptotic behavior of random pseudomatrices for two sequences of states: the sequence of distinguished states, with respect to which our variables will be matricially free, and the sequence of traces which are normalized sums of conditions in the definition of matricial freeness. Under suitable assumptions, we will later obtain two types of central limit theorems: the standard central limit theorem as well as the tracial central limit theorem for matricially free random variables, the latter being related to random matrix models.

Let $(\mathcal{A}_n)_{n \in \mathbb{N}}$ be a sequence of unital $*$ -algebras. For each n , let $(X_{i,j}(n))_{1 \leq i,j \leq n}$ be an array of self-adjoint random variables in \mathcal{A}_n and let $(\phi_{i,j}(n))$ be an array of associated states on \mathcal{A}_n . We will say that the moments of the $X_{i,j}(n)$ are *uniformly bounded* with respect to $(\phi_{i,j}(n))$ if, for all natural m , there exists $M_m \geq 0$ such that

$$(6.1) \quad |\phi_{i,j}(n)(X_{i,j}^m(n))| \leq \frac{M_m}{n^{m/2}}$$

for all $n \in \mathbb{N}$ and $i, j \in [n]$.

Assume further that for each natural n the array $(X_{i,j}(n))$ is matricially free with respect to the array $(\phi_{i,j}(n))$ defined by a distinguished state ϕ_n and the conditions $\psi_{n,j}$, where $1 \leq j \leq n$. We are going to study the asymptotic behavior of random pseudomatrices

$$(6.2) \quad S(n) = \sum_{i,j=1}^n X_{i,j}(n)$$

with respect to two types of states:

- (1) distinguished states ϕ_n ,
- (2) convex linear combinations of conditions

$$(6.3) \quad \psi_n := \frac{1}{n} \sum_{j=1}^n \psi_{n,j}$$

which play the role of traces in the context of random pseudomatrices.

This will lead to two types of central limit theorems for matricially free random variables: the ‘standard CLT’ reminding the CLT for free random variables [20] and the ‘tracial CLT’ reminding the limit theorem for random matrices [22]. Thanks to the ‘matricial property’ and the ‘diagonal subordination property’ of the matricially free product of states, the normalization of square root type works in both cases since the number of summands in $S(n)$ which give a non-zero contribution to the limits is in both cases of order n .

Of course, the uniform boundedness assumption is satisfied if we take variables of type $X_{i,j}(n) = X_{i,j}/\sqrt{n}$, where $(X_{i,j})$ is an infinite array of random variables whose distributions in the states $\phi_{i,j}(n)$, respectively, are identical and do not depend on n . Although it is convenient to think of the $X_{i,j}(n)$ as if they were of this form, we want to study similar variables, whose variances are of order $1/n$ and stay the same within blocks whose sizes become infinite as $n \rightarrow \infty$.

If the tuple $((i_1, j_1), \dots, (i_m, j_m)) \in (I \times I)^m$, where I is an index set, defines a partition $\pi = \{\pi_1, \dots, \pi_k\}$ of the set $[m]$, i.e. $(i_p, j_p) = (i_q, j_q)$ if and only if there exists r such that $p, q \in \pi_r$, we will write

$$P((i_1, j_1), \dots, (i_m, j_m)) = \pi.$$

Of course, if π is a non-crossing pair partition, then each π_r is a two-element set. We will also adopt the convention that if m is odd, then $\mathcal{NC}_m^2 = \emptyset$ and the summation over $\pi \in \mathcal{NC}_m^2$ gives zero. This allows us to state results for moments of the $S(n)$ of all orders without distinguishing even and odd moments.

Lemma 6.1. *Let (\mathcal{A}_n, ϕ_n) be a sequence noncommutative probability spaces, each with an array $(\phi_{i,j}(n))$ defined as above and such that*

- (1) *for fixed $n \in \mathbb{N}$, the array $(X_{i,j}(n))$ is matricially free with respect to $(\phi_{i,j}(n))$,*
- (2) *$\phi_{i,j}(n)(X_{i,j}(n)) = 0$ and $\phi_{i,j}(n)(X_{i,j}^2(n)) = v_{i,j}(n)$ for any i, j and $n \in \mathbb{N}$,*
- (3) *the moments of the $X_{i,j}(n)$ are uniformly bounded with respect to $(\phi_{i,j}(n))$.*

Then

$$\psi_n(S^m(n)) = \sum_{\pi \in \mathcal{NC}_m^2} v(\pi, n) + O\left(\frac{1}{\sqrt{n}}\right)$$

where $v(\pi, n) = \text{tr}(V(\pi, n))$ is given by Definition 5.1 and corresponds to the variance matrix $V(n) = (v_{i,j}(n))$.

Proof. We have

$$\begin{aligned} \psi_n(S^m(n)) &= \sum_{i_1, j_1, \dots, i_m, j_m} \psi_n(X_{i_1, j_1}(n) \dots X_{i_m, j_m}(n)) \\ &= \sum_{\pi \in \mathcal{P}_m} \sum_{\substack{i_1, j_1, \dots, i_m, j_m \\ P((i_1, j_1), \dots, (i_m, j_m)) = \pi}} \psi_n(X_{i_1, j_1}(n) \dots X_{i_m, j_m}(n)) \end{aligned}$$

where \mathcal{P}_m denotes the set of all partitions of $[m]$. Standard arguments allow us to conclude that for large n only non-crossing pair partitions give relevant contributions, namely

- (1) the moments of the $X_{i,j}(n)$ satisfy the singleton condition with respect to the $\psi_{n,k}$ and thus with respect to ψ_n , namely

$$\psi_n(X_{i_1, j_1}(n) \dots X_{i_m, j_m}(n)) = 0$$

if $\pi = P((i_1, j_1), \dots, (i_m, j_m))$ contains a singleton,

- (2) if π has no singletons and contains blocks consisting of more than two elements, then

$$\sum_{\substack{i_1, j_1, \dots, i_m, j_m \\ P((i_1, j_1), \dots, (i_m, j_m)) = \pi}} \psi_n(X_{i_1, j_1}(n) \dots X_{i_m, j_m}(n)) = O\left(\frac{1}{\sqrt{n}}\right)$$

- (3) if m is even and π is a crossing pair partition of $[m]$, then the contribution from the corresponding mixed moments of the $X_{i,j}$ is zero.

The above arguments are similar to those that would hold if random variables $X_{i,j}(n)$, where $(i, j) \in I \times I$, were free. (1) follows from the definition of the matricially free product of states and the mean zero assumption. (2) holds since moments are uniformly bounded and the usual combinatorial argument (there are fewer than $m/2$ independent indices to sum over) can be used. In particular, this and (1) imply that if m is odd, then the contribution to the limit is of order $O(1/\sqrt{n})$. (3) is a consequence of the mean zero assumption and the definition of the matricial freeness (the mixed moments in our case vanish always when the corresponding moments vanish in the free case).

Therefore,

$$\psi_n(S^m(n)) = \sum_{\pi \in \mathcal{NC}_m^2} \sum_{\substack{i_1, j_1, \dots, i_m, j_m \\ P((i_1, j_1), \dots, (i_m, j_m)) = \pi}} \psi_n(X_{i_1, j_1}(n) \dots X_{i_m, j_m}(n)) + O\left(\frac{1}{\sqrt{n}}\right).$$

Now, suppose that m is even, $\pi \in \mathcal{NC}_m^2$ and the sequence of pairs $((i_1, j_1), \dots, (i_m, j_m))$ is ‘compatible with the matricial multiplication’, i.e. such that if (i_k, j_k) and (i_r, j_r) label an inner-outer pair of blocks, then $j_k = i_r$. If $\pi_j = \{r, r+1\}$ is a block which has no inner blocks, then we can ‘pull out’ the variance corresponding to that block, namely:

$$\psi_n(X_{i_1, j_1}(n) \dots X_{i_m, j_m}(n))$$

$$= v_{i_r, j_r} \psi_n(X_{i_1, j_1}(n) \dots X_{i_{r-1}, j_{r-1}}(n) X_{i_{r+2}, j_{r+2}}(n) \dots X_{i_m, j_m}(n))$$

(mean zero assumption is used again). Continuing this procedure with other blocks which have no inner blocks, and summing over indices $i_1, j_1, \dots, i_m, j_m$, for which it holds that $P((i_1, j_1), \dots, (i_m, j_m)) = \pi$, we arrive at

$$\frac{1}{n} \sum_{k_0, k_1, \dots, k_m} v_{k_1, k_{\sigma(1)}}(n) \dots v_{k_m, k_{\sigma(m)}}(n) + O\left(\frac{1}{\sqrt{n}}\right)$$

where $\sigma(r) = 0$ if π_r has no outer blocks (k_0 labels the conditional block) and $\sigma(r) = j$ if the nearest outer block of π_r is labelled by j . Let us observe that we included in the above sum all possible labellings of the blocks of π . This is done for convenience since it enables us to express the final result in terms of $v(\pi)$. More explicitly, we allow k_0, k_1, \dots, k_m to assume arbitrary values from the set $[m]$ (in particular, they can all be equal), which produces certain terms which cannot be obtained from the summation over all $((i_1, j_1), \dots, (i_m, j_m))$ which define π . For example, no nearest inner-outer pair of blocks can contribute $v_{j,j} v_{j,j}$, which appears in the above sum. However, all such terms are of order $1/\sqrt{n}$ due to insufficient number of different summation indices (there are fewer than $m/2$ independent indices) and therefore they can be included in the sum without changing the asymptotics. Using Lemma 5.1, we obtain our assertion. \blacksquare

Lemma 6.2. *Under the assumptions of Lemma 6.1 it holds that*

$$\phi_n(S^m(n)) = \sum_{\pi \in \mathcal{NC}_m^2} v_0(\pi, n) + O\left(\frac{1}{\sqrt{n}}\right)$$

where $v_0(\pi, n) = \text{Tr}(V_0(\pi, n))$ is given by Definition 5.2 and corresponds to the variance matrix $V(n) = (v_{i,j}(n))$.

Proof. The proof is similar to that of Lemma 6.1. ■

In order to ensure existence of the limits of $v(\pi, n)$ and $v_0(\pi, n)$ as $n \rightarrow \infty$, we need to make some assumptions on the sequence of matrices $(V(n))_{n \in \mathbb{N}}$. For that purpose, introduce a matrix, called the *dimension matrix*

$$D = \text{diag}(d_1, d_2, \dots, d_r) \in D_r(\mathbb{R}) \quad \text{with} \quad \text{Tr}(D) = 1,$$

where d_1, d_2, \dots, d_r are positive real numbers. For given natural n , we associate with D the partition $[n] = N_1 \cup N_2 \cup \dots \cup N_r$, where

$$N_1 = [1, n_1], N_2 = [1 + n_1, n_1 + n_2], \dots, N_r = [1 + n_1 + \dots + n_{r-1}, n]$$

and

$$n_k = E \left(\sum_{i=1}^k d_i n \right) - E \left(\sum_{i=1}^{k-1} d_i n \right)$$

for each $k \in [r]$, with $E(x)$ denoting the largest integer smaller or equal to x . Of course, the N_k are disjoint and naturally ordered intervals containing n_k natural numbers, respectively. Note that in the limit $n \rightarrow \infty$ we obtain $n_k/n \rightarrow d_k$ for each k .

Assume now that each matrix $V(n)$ has the block-form

$$(6.4) \quad V(n) = \begin{pmatrix} V_{1,1}(n) & \dots & V_{1,r}(n) \\ \vdots & \ddots & \vdots \\ V_{r,1}(n) & \dots & V_{r,r}(n) \end{pmatrix}$$

where each block $V_{i,j}(n)$ consists of the same number $u_{i,j}/n$. In other words, each $V(n)$ is obtained from a real-valued matrix

$$U = (u_{i,j}) \in M_r(\mathbb{R})$$

by repeating $n_i \times n_j$ times each entry $u_{i,j}$ at all entries of block $V_{i,j}(n)$ and dividing it by n . Unless stated otherwise, we assume that $u_{j,j} > 0$ for any j and that $u_{i,j} \geq 0$ for any $i \neq j$. The whole sequence $(V(n))_{n \in \mathbb{N}}$ is built in such a way that proportions between sizes of these blocks are similar for all n and expressed in terms of the dimension matrix D (asymptotically, these proportions are given by the proportions between numbers d_j).

Assuming that the variance matrices $V(n)$ are of the above block form, we can now state the standard and tracial central limit theorems, with limit distributions described in terms of traces of Section 5.

Theorem 6.1. *Under the assumptions of Lemma 6.1, if $V(n)$ is of the block form (6.4) for each $n \in \mathbb{N}$, then*

$$(6.5) \quad \lim_{n \rightarrow \infty} \psi_n(S^m(n)) = \sum_{\pi \in \mathcal{NC}_m^2} b(\pi),$$

for any $m \in \mathbb{N}$, where $b(\pi) = \text{Tr}(B(\pi)D)$ and $B(\pi)$ is the diagonal matrix of Definition 5.1 corresponding to π and the matrix $B = DU$.

Proof. Clearly, if m is odd, we get zeros on both sides of the above formula (we use our convention that in this case $\mathcal{NC}_m^2 = \emptyset$). The proof for $m = 2k$, where $k \in \mathbb{N}$, is based on Lemmas 5.1 and 6.1. If, in the combinatorial expression for $v(\pi, n)$, we

substitute for the matrix elements of $V(n)$ the assumed block form, then, using the partition of the set of colors $[n] = N_1 \cup N_2 \cup \dots \cup N_r$, we can perform summations over the colorings which belong to each interval N_j separately. Thus, the contributions of various $[n]$ -colorings of π to the limit laws reduce to those corresponding to $[r]$ -colorings and are described in terms of numbers $u_j(\pi_i, f)$, where $i \in [k]$, $j \in [r]$ and $f \in F_r(\pi)$ (the number j is the color of the conditional block). We have

$$\begin{aligned} \lim_{n \rightarrow \infty} v(\pi, n) &= \lim_{n \rightarrow \infty} \left(\frac{1}{n^{k+1}} \sum_{j \in [r]} n_j \sum_{f \in F_r(\pi)} n_{f(1)} u_j(\pi_1, f) \dots n_{f(k)} u_j(\pi_k, f) \right) \\ &= \sum_{j \in [r]} d_j \sum_{f \in F_r(\pi)} d_{f(1)} u_j(\pi_1, f) \dots d_{f(k)} u_j(\pi_k, f) = \text{Tr}(B(\pi)D) \end{aligned}$$

where $\pi \rightarrow B(\pi)$ is the matrix-valued function which corresponds to the matrix $B = DU$ in accordance with Definition 5.1. In terms of matrix multiplication, the expression on the right hand side is obtained from that of Lemma 5.1 corresponding to matrix U by multiplying U from the left by the dimension matrix D and multiplying the whole product of matrices from the right by D . This proves our assertion. \blacksquare

Theorem 6.2. *Under the assumptions of Lemma 6.2, if $V(n)$ is of the block form (6.4) for each $n \in \mathbb{N}$, then*

$$(6.6) \quad \lim_{n \rightarrow \infty} \phi_n(S^m(n)) = \sum_{\pi \in \mathcal{NC}_m^2} b_0(\pi),$$

for any $m \in \mathbb{N}$, where $\pi \rightarrow b_0(\pi)$ is the real-valued function of Definition 5.2 corresponding to the matrix $B = DU$.

Proof. The proof is similar to that of Theorem 6.1 and is based on Lemmas 5.1 and 6.2. The only difference is that we do not use conditional blocks to describe the colorings of all blocks of π . Thus, the contributions of various $[n]$ -colorings of π to the limit laws reduce to those corresponding to $[r]$ -colorings and are described in terms of numbers $u(\pi_i, f)$, where $i \in [k]$ and $f \in F_r(\pi)$. Namely, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} v_0(\pi, n) &= \lim_{n \rightarrow \infty} \left(\frac{1}{n^k} \sum_{f \in F_r(\pi)} n_{f_1} u(\pi_1, f) \dots n_{f_k} u(\pi_k, f) \right) \\ &= \sum_{f \in F_r(\pi)} d_{f_1} u(\pi_1, f) \dots d_{f_k} u(\pi_k, f) = \text{Tr}(B_0(\pi)) \end{aligned}$$

as $n \rightarrow \infty$, where $\pi \rightarrow B_0(\pi)$ is the function defined by Definition 5.2, which proves our assertion. \blacksquare

7. MATRICIAL SEMICIRCLE DISTRIBUTIONS

The results of Section 6 lead to combinatorial formulas for the asymptotic moments in the corresponding central limit theorems. In this Section we are going to express the limits in terms of their Cauchy transforms represented in the form of continued fractions. They play the role of the (standard and tracial) ‘matricial semicircle distributions’.

For that purpose let us recall definitions of certain convolutions of distributions, or more generally, of probability measures. If F_μ is the reciprocal Cauchy transform of some probability measure $\mu \in \mathcal{M}_\mathbb{R}$, then the K-transform of μ is given by $K_\mu(z) = z - F_\mu(z)$. The boolean additive convolution $\mu \uplus \nu$ can be defined by the equation

$$K_{\mu \uplus \nu}(z) = K_\mu(z) + K_\nu(z)$$

where $\mu, \nu \in \mathcal{M}_\mathbb{R}$ and $z \in \mathbb{C}^+$, respectively. In fact, this equation shows that the K-transform is the boolean analog of the logarithm of the Fourier transform [19].

We will also need another convolution, which reminds the monotone convolution [17], called the orthogonal additive convolution and defined by the equation

$$K_{\mu \vdash \nu}(z) = K_\mu(F_\nu(z))$$

where $\mu, \nu \in \mathcal{M}_\mathbb{R}$ and $z \in \mathbb{C}^+$. It was introduced in [12], where we showed that the above formula defines a unique probability measure on the real line. Moreover, if μ and ν are compactly supported, both $\mu \vdash \nu$ and $\mu \uplus \nu$ are compactly supported.

Using these convolutions, we will now define certain important continued fractions which converge uniformly on the compact subsets of \mathbb{C}^+ to the K-transforms of some probability measures $\mu_{i,j} \in \mathcal{M}_\mathbb{R}$.

Lemma 7.1. *For given $B \in M_r(\mathbb{R})$ with nonnegative entries, continued fractions of the form*

$$K_{i,j}(z) = \frac{b_{i,j}}{z - \sum_k \frac{b_{k,i}}{z - \sum_p \frac{b_{p,k}}{z - \dots}}}$$

where $i, j \in [r]$, converge uniformly on the compact subsets of \mathbb{C}^+ to the K-transforms of some $\mu_{i,j} \in \mathcal{M}_\mathbb{R}$ with compact supports.

Proof. Let us define a sequence of functions which approximate the $K_{i,j}$. Namely, set $K_{i,j}^{(0)}(z) = b_{i,j}/z$ for any $i, j \in [r]$, which are the K-transforms of probability measures on \mathbb{R} for any i, j (Bernoulli measures if $b_{i,j} > 0$ and δ_0 if $b_{i,j} = 0$). In order to use an inductive argument, let us establish the recurrence

$$K_{i,j}^{(m)}(z) = \frac{b_{i,j}}{z - \sum_p K_{p,i}^{(m-1)}(z)}$$

for $m \geq 1$. If the $K_{p,i}^{(m-1)}$ are the K-transforms of some $\mu_{p,i}^{(m-1)} \in \mathcal{M}_\mathbb{R}$ for any i and p , respectively, then the sums $\sum_p K_{p,i}^{(m-1)}$ are the K-transforms of some

$$\mu_i^{(m-1)} = \mu_{1,i}^{(m-1)} \uplus \mu_{2,i}^{(m-1)} \uplus \dots \uplus \mu_{r,i}^{(m-1)} \in \mathcal{M}_\mathbb{R},$$

and next, the $K_{i,j}^{(m)}$ are the K-transforms of some

$$\mu_{i,j}^{(m)} = \kappa_{i,j} \vdash \mu_i^{(m-1)} \in \mathcal{M}_\mathbb{R}$$

where the $\kappa_{i,j}$ are the Bernoulli measures with K-transforms $K_{i,j}(z) = b_{i,j}/z$, respectively. It is easy to see that all these measures are compactly supported. Moreover, the properties of the orthogonal additive convolution (Corollary 5.3 in [10]) say that the

moments of $\mu_{i,j}^{(n)}$ of orders $\leq 2m$ agree with the corresponding moments of $\mu_{i,j}^{(m)}$ for any $n > m$ and any given i, j . Therefore, we have weak convergence

$$\text{w-} \lim_{m \rightarrow \infty} \mu_{i,j}^{(m)} = \mu_{i,j}$$

to some $\mu_{i,j} \in \mathcal{M}_{\mathbb{R}}$ for any i, j . These measures are also compactly supported since $\sup_{i,j} b_{i,j}$ is finite. In turn, this implies that the corresponding Cauchy transforms (and thus K-transforms) converge uniformly to the Cauchy transform (K-transforms) of $\mu_{i,j}$ on compact subsets of \mathbb{C}^+ . This completes the proof. \blacksquare

We are ready to state a theorem, which can be viewed as the tracial central limit theorem for matricially free random variables.

Theorem 7.1. *Under the assumptions of Theorem 6.1, the ψ_n -distributions of S_n converge weakly to the distribution given by the convex linear combination*

$$\mu = \sum_{j=1}^r d_j \mu_j$$

where $\mu_j = \mu_{1,j} \uplus \mu_{2,j} \uplus \dots \uplus \mu_{r,j}$ for each $j = 1, \dots, r$ and $\mu_{i,j}$ is the distribution defined by $K_{i,j}$ for any i, j .

Proof. By Theorem 6.1, we have combinatorial formulas for the moments M_m of the limit law in the tracial central limit theorem. The associated distribution extends to a unique compactly supported probability measure μ on the real line since its moments are bounded by the moments of the Wigner semicircle distribution σ_a with variance $a = \sup_{i,j} b_{i,j}$. Using the multiplicative formula (5.4) for $B(\pi)$, we can formally write the Cauchy transform of μ in the form

$$\begin{aligned} G_{\mu}(z) &= \sum_{k=0}^{\infty} M_{2k} z^{-2k-1} \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \left(\sum_{\pi \in \mathcal{NC}_{2k}^2} \text{Tr}(B(\pi)D) \right) z^{-2k-1} \\ &= \text{Tr} \left((z - K(z))^{-1} D \right), \end{aligned}$$

which can be called the ‘trace formula’ for G_{μ} , where

$$(7.1) \quad K(z) = \sum_{k=1}^{\infty} \sum_{\pi \in \mathcal{NCC}_{2k}^2} B(\pi) z^{-2k+1}$$

is a diagonal-matrix-valued formal power series. Moreover, we will show below that each function K_j on its diagonal is, in fact, the K-transform of some $\mu_j \in \mathcal{M}_{\mathbb{R}}$. Then, the formal power series given by the trace formula is the Cauchy transform of μ as a convex linear combination of Cauchy transforms of probability measures. In fact, using the definition of $B(\pi)$ and (5.3), we obtain the equation

$$(7.2) \quad K(z) = \tau((z - K(z))^{-1} B)$$

where $K(z) = \text{diag}(K_1(z), \dots, K_r(z))$. By analogy with the scalar-valued case, we can find its solution in the form of a continued fraction. Namely, observe that each $K_j(z)$ has the form of a formal Laurent series

$$\frac{c_{-1}}{z} + \frac{c_{-3}}{z^3} + \dots$$

for some c_{-1}, c_{-3}, \dots , and therefore the above vector equation can be solved by successive approximations. Namely, we set μ_j to be the (compactly supported) probability measure associated with the K-transform $K_j(z) = \sum_i K_{i,j}(z)$ for each $j \in [r]$, where the $K_{i,j}$ are given by Lemma 7.1 for $B = DU$. These K-transforms solve (7.2). This, together with the trace formula for G_μ , gives

$$G_\mu = \sum_{j=1}^r d_j G_{\mu_j}$$

where $G_{\mu_j}(z) = 1/(z - K_j(z))$ is the Cauchy transform of μ_j for $j = 1, \dots, r$. That completes the proof. \blacksquare

Remark 7.1. The Cauchy transform of each μ_j can be written as a continued fraction of the form

$$G_{\mu_j}(z) = \frac{1}{z - \sum_i \frac{b_{i,j}}{z - \sum_k \frac{b_{k,i}}{z - \sum_p \frac{b_{p,k}}{z - \dots}}}}$$

which converges on the compact subsets of \mathbb{C}^+ .

Next, we state a theorem, which plays the role of the standard central limit theorem for matricially free random variables.

Theorem 7.2. *Under the assumptions of Theorem 6.2, the Φ_n -distributions of S_n converge weakly to the distribution*

$$\mu_0 = \mu_{1,1} \uplus \mu_{2,2} \uplus \dots \uplus \mu_{r,r}$$

where $\mu_{j,j}$ is the distribution defined by $K_{j,j}$ for each j .

Proof. By Theorem 6.2, we have combinatorial expressions for the limit moments M_m . The proof is similar to that of Theorem 7.1 and is based on the trace formula for the K-transform of μ_0

$$K_{\mu_0}(z) = \text{Tr}((z - K(z))^{-1} B_0)$$

derived from the definition of the function b_0 , which leads to the equation for the Cauchy transform

$$G_{\mu_0}(z) = \frac{1}{z - \sum_j K_{j,j}(z)},$$

which completes the proof. \blacksquare

Remark 7.2. The Cauchy transform G_{μ_0} can be written as a continued fraction of the form

$$G_{\mu_0}(z) = \frac{1}{z - \sum_j \frac{b_{j,j}}{z - \sum_i \frac{b_{i,j}}{z - \sum_k \frac{b_{k,i}}{z - \dots}}}}$$

which gives a matricial extension of the continued fraction of the Wigner semicircle distribution.

8. DECOMPOSITIONS IN TERMS OF SUBORDINATIONS

In the one-dimensional case the limit distributions μ_0 and μ are related by Proposition 5.1. In particular, if each variance matrix $V(n)$ has identical entries equal to one, both central limit theorems (standard and tracial) give the Wigner semicircle distribution with variance 1 (of course, the standard case also follows from free probability, whereas the tracial case is related to random matrices).

In this Section we will analyze in more detail the limit distributions for the two-dimensional case, namely when each variance matrix $V(n)$ consists of four blocks. They will be expressed in terms of two-dimensional arrays of distributions. Finding simple analytic formulas for the corresponding four-parameter Cauchy transforms and densities does not seem possible in the general case. However, we shall derive decomposition formulas for those measures in terms of s-free additive convolutions [12], which gives some insight into their structure (see also [18] for recent results on the multivariate case). The s-free additive convolution refers to the subordination property for free additive convolution, discovered by Voiculescu [24] and generalized by Biane [4]. As shown in [12] and [13], there is a notion of independence, called *freeness with subordination*, or simply *s-freeness*, associated with the s-free additive convolution and its multiplicative counterpart.

Recall that the s-free additive convolution of $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ is the unique probability measure $\mu \boxplus \nu \in \mathcal{M}_{\mathbb{R}}$ defined by the subordination equation

$$\nu \boxplus \mu = \nu \triangleright (\mu \boxplus \nu),$$

where \triangleright denotes the monotone additive convolution [17]. Equivalently, the above subordination property can be written in terms of Cauchy transforms or their reciprocals.

Using s-free additive convolutions and the boolean convolution, we obtain a decomposition of the free additive convolution of the form

$$\mu \boxplus \nu = (\mu \boxplus \nu) \uplus (\nu \boxplus \mu),$$

which allows us to interpret both s-free additive convolutions appearing here as (in general, non-symmetric) halves of $\mu \boxplus \nu$. We find it interesting that the limit distributions in the two-dimensional case will turn out to be deformations of the free additive convolution of semicircle laws implemented by this decomposition. In other words, the subordination property and the associated convolutions give a natural framework for studying matricial generalizations of the semicircle law.

For simplicity, it will be convenient to use the indices-free notation for the two-dimensional matrix of K-transforms:

$$\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = \begin{pmatrix} K_{1,1}(z) & K_{1,2}(z) \\ K_{2,1}(z) & K_{2,2}(z) \end{pmatrix}$$

and

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt{b_{1,1}} & \sqrt{b_{1,2}} \\ \sqrt{b_{2,1}} & \sqrt{b_{2,2}} \end{pmatrix} = \sqrt{B}$$

where the square root is interpreted entry-wise.

Moreover, we will distinguish two laws by special notations: we denote by σ_α the Wigner semicircle distribution with the Cauchy transform

$$G_{\sigma_\alpha}(z) = \frac{z - \sqrt{z^2 - 4\alpha^2}}{2\alpha^2},$$

where the branch of $\sqrt{z^2 - 4\alpha^2}$ is chosen so that $\sqrt{z^2 - 4\alpha^2} > 0$ if $z \in \mathbb{R}$ and $z \in (2\alpha, \infty)$, and by κ_γ we denote the Bernoulli law with the Cauchy transform

$$G_{\kappa_\gamma}(z) = \frac{1}{z - \gamma^2/z},$$

i.e. $\sigma_\gamma = 1/2(\delta_{-\gamma} + \delta_\gamma)$.

Finally, we will also use the *boolean compressions* of $\mu \in \mathcal{M}_\mathbb{R}$, where $t \geq 0$, defined by multiplying its K-transform by t , namely we define $T_t\mu$ to be the (unique) probability measure on \mathbb{R} , for which

$$K_{T_t\mu} = tK_\mu.$$

These transformations were introduced and studied in [7] and called ‘ t -transformations’ of μ . We allow $t = 0$, in which case $T_0\mu = \delta_0$. In particular, we shall use two-parameter boolean compressions of semicircle distributions, $\sigma_{\alpha,\beta} = T_t\sigma_\alpha$ for $t = (\beta/\alpha)^2$, with

$$G_{\sigma_{\alpha,\beta}}(z) = \frac{(2\alpha^2 - \beta^2)z - \beta^2\sqrt{z^2 - 4\alpha^2}}{(2\alpha^2 - 2\beta^2)z^2 + 2\beta^4}$$

being their Cauchy transforms, where the branch of the square root is the same as in the case of G_{σ_α} .

Theorem 8.1. *If $\alpha, \beta, \gamma, \delta \neq 0$, then the diagonal measures $\mu_{j,j}$ defined by $K_{j,j}$, where $1 \leq j \leq 2$, have the form*

$$\begin{aligned} \mu_{1,1} &= T_{1/t}(\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma}) \\ \mu_{2,2} &= T_{1/s}(\sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}) \end{aligned}$$

with the non-diagonal measures given by $\mu_{1,2} = T_t\mu_{1,1}$ and $\mu_{2,1} = T_s\mu_{2,2}$, where $t = (\beta/\alpha)^2$ and $s = (\gamma/\delta)^2$,

Proof. It is easy to see that the following algebraic relations hold:

$$\begin{aligned} a(z) &= \frac{\alpha^2}{z - a(z) - c(z)} \quad \text{and} \quad d(z) = \frac{\delta^2}{z - b(z) - d(z)}, \\ b(z) &= \frac{\beta^2}{z - a(z) - c(z)} \quad \text{and} \quad c(z) = \frac{\gamma^2}{z - b(z) - d(z)}. \end{aligned}$$

Thus, $b(z) = ta(z)$ and $c(z) = sd(z)$, which gives $\mu_{1,2} = T_t\mu_{1,1}$ and $\mu_{2,1} = T_s\mu_{2,2}$. In turn, from the equation for $a(z)$, we get

$$a(z) = \frac{z - c(z) - \sqrt{(z - c(z))^2 - 4\alpha^2}}{2} = K_{\sigma_\alpha}(z - c(z))$$

and thus $\mu_{1,1} = \sigma_\alpha \vdash \mu_{2,1}$. In a similar manner we obtain $\mu_{2,2} = \sigma_\delta \vdash \mu_{1,2}$. Therefore, we arrive at the equations

$$\begin{aligned}\mu_{1,1} &= \sigma_\alpha \vdash (T_s\sigma_\delta \vdash T_t\mu_{1,1}) \\ \mu_{2,2} &= \sigma_\delta \vdash (T_t\sigma_\alpha \vdash T_s\mu_{2,2})\end{aligned}$$

since $T_t(\mu \vdash \nu) = (T_t\mu) \vdash \nu$. In order to express the $\mu_{j,j}$ in terms of s-free additive convolutions, we need to use the properties of the orthogonal convolution. We have shown in [12] that the moment of order k of $\mu \vdash \nu$ depends on the moments of orders $\leq k$ of μ and the moments of orders $\leq k-2$ of ν . This leads to the conclusion that for any compactly supported $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ and the associated sequence of measures $(\mu \vdash_m \nu)$, defined recursively by

$$\mu \vdash_m \nu = \mu \vdash (\nu \vdash_{m-1} \mu) \quad \text{with} \quad \mu \vdash_1 \nu = \mu \vdash \nu,$$

we have weak convergence $w - \lim_{m \rightarrow \infty} \mu \vdash_m \nu = \mu \boxplus \nu$. If we take $\mu = T_t\omega_\alpha$ and $\nu = T_s\omega_\delta$ (these measures are compactly supported), we get the desired formulas. ■

Corollary 8.1. *The measures μ_0, μ_1, μ_2 can be decomposed as*

$$\begin{aligned}\mu_0 &= T_{1/t}(\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma}) \uplus T_{1/s}(\sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}) \\ \mu_1 &= T_{1/t}(\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma}) \uplus (\sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}) \\ \mu_2 &= T_{1/s}(\sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}) \uplus (\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma})\end{aligned}$$

where the assumptions and notations are the same as in Theorem 8.1.

Proof. These decompositions follow immediately from Theorems 7.1, 7.2 and 8.1. ■

Remark 8.1. The formulas for the diagonal measures $\mu_{j,j}$ in the proof of Theorem 8.1 remind those for 2-periodic continued fractions if we take $t = s = 1$. The latter are of the same form, except that the semicircle distributions are replaced by much simpler Bernoulli laws. Nevertheless, if $t = s = 1$, the formulas for the μ_j take a simple form

$$\mu_j = \sigma_\alpha \boxplus \sigma_\delta$$

where $j = 0, 1, 2$. By Theorem 7.1 and Corollary 8.1, the same formula holds for μ . Therefore, all measures μ_0, μ_1, μ_2, μ can be viewed as deformations of the free additive convolution of two semicircle distributions, implemented by means of boolean compressions.

Let us consider now the situation in which some of the numbers $\alpha, \beta, \gamma, \delta$ vanish. Suitable formulas can be derived algebraically, as we did in the proof of Theorem 8.1. However, one can also obtain the same results by taking weak limits in the formulas for the measures $\mu_{i,j}$, using the fact that all measures involved have compact supports. For that purpose, let us state a few useful facts about weak limits which will be of

interest to us. Then, we consider eight cases, to which the remaining cases are similar (for instance, $\alpha = \beta = 0$ is similar to $\gamma = \delta = 0$).

Proposition 8.1. *Let $t = \beta^2/\alpha^2$, where $\alpha, \beta > 0$, and let $\mu \in \mathcal{M}_{\mathbb{R}}$ be compactly supported.*

- (1) *If $\alpha \rightarrow 0^+$, then*
 - (a) $w - \lim \sigma_{\alpha} = \delta_0$,
 - (b) $w - \lim (\sigma_{\alpha, \beta}) = \kappa_{\beta}$,
 - (c) $w - \lim (T_{1/t}(\sigma_{\alpha, \beta} \boxplus \mu)) = \delta_0$.
- (2) *If $\beta \rightarrow 0^+$, then*
 - (a) $w - \lim \sigma_{\alpha, \beta} = \delta_0$,
 - (b) $w - \lim (T_t \mu) = \delta_0$,
 - (c) $w - \lim (T_{1/t}(\sigma_{\alpha, \beta} \boxplus \mu)) = \sigma_{\alpha} \vdash \mu$.

Proof. If $\alpha \rightarrow 0^+$, then $K_{\sigma_{\alpha}}(z) \rightarrow 0$, which proves (1a). Here, as well as in the remaining cases, convergence is uniform on compact subsets of \mathbb{C}^+ . Moreover, $K_{\sigma_{\alpha, \beta}} = \beta^2/(z - \alpha^2 K_{\sigma_{\alpha}}) \rightarrow \beta^2/z = K_{\kappa_{\beta}}$, which gives (1b). In turn, if $\beta \rightarrow 0^+$, then $K_{\sigma_{\alpha, \beta}}(z) = \beta^2/(z - K_{\sigma_{\alpha}}(z)) \rightarrow 0$, thus also $w - \lim \sigma_{\alpha, \beta} = \delta_0$, which proves (2a). If, in addition $t \rightarrow 0^+$, then $T_t \mu \rightarrow \delta_0$ for any $\mu \in \mathcal{M}_{\mathbb{R}}$, which proves (2b). Finally,

$$T_{1/t}(\sigma_{\alpha, \beta} \boxplus \mu) = T_{1/t}(T_t \sigma_{\alpha} \vdash (\mu \boxplus \sigma_{\alpha, \beta})) = \sigma_{\alpha} \vdash (\mu \boxplus \sigma_{\alpha, \beta})$$

and thus the right hand side tends weakly to δ_0 as $\alpha \rightarrow 0^+$, which gives (1c), and tends weakly to $\sigma_{\alpha} \vdash (\mu \boxplus \delta_0) = \sigma_{\alpha} \vdash \mu$, which gives (2c). This holds for any $\mu \in \mathcal{M}_{\mathbb{R}}$, and we also use the right unit property of δ with respect to the s-free additive convolution, namely $\mu \boxplus \delta_0 = \mu$. ■

Corollary 8.2. *If some of the entries of the matrix A vanish, we can distinguish eight different cases, for which the distributions $\mu_{i,j}$ are given by Table 1.*

Proof. If $a_{i,j} = 0$, then $\mu_{i,j}$ is the Dirac delta, which easily follows from the algebraic equations for the corresponding K-transforms. An alternative proof can be given by taking weak limits of the formulas of Theorem 8.1, as we proceed with the remaining measures. Thus, if $\delta \rightarrow 0^+$, then

$$\begin{aligned} \mu_{1,1} &= w - \lim T_{1/t}(\sigma_{\alpha, \beta} \boxplus \sigma_{\delta, \gamma}) = T_{1/t}(\sigma_{\alpha, \beta} \boxplus \kappa_{\gamma}) \\ \mu_{1,2} &= w - \lim (\sigma_{\alpha, \beta} \boxplus \sigma_{\delta, \gamma}) = \sigma_{\alpha, \beta} \boxplus \kappa_{\gamma}, \\ \mu_{2,1} &= w - \lim (\sigma_{\delta, \gamma} \boxplus \sigma_{\alpha, \beta}) = \kappa_{\gamma} \boxplus \sigma_{\alpha, \beta}, \end{aligned}$$

by (1b) of Proposition 8.1, which proves the first case in Table 1. The remaining cases are proved in a similar manner. ■

In the case of arbitrary matrix A , finding the four-parameter densities of μ_0 and μ is unwieldy. Below we shall just consider two special cases, in which we can find nice formulas for these measures for matrices A of arbitrary dimension. These two cases are of special interest since they are associated with (asymptotic) freeness and (asymptotic) monotone independence.

TABLE 1. Distributions $\mu_{i,j}$ in the case A has zero entries

$a_{1,1}$	$a_{1,2}$	$a_{2,1}$	$a_{2,2}$	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{2,1}$	$\mu_{2,2}$
α	β	γ	0	$T_{1/t}(\sigma_{\alpha,\beta} \boxplus \kappa_\gamma)$	$\sigma_{\alpha,\beta} \boxplus \kappa_\gamma$	$\kappa_\gamma \boxplus \sigma_{\alpha,\beta}$	δ_0
0	β	γ	0	δ_0	$\kappa_\beta \boxplus \kappa_\gamma$	$\kappa_\gamma \boxplus \kappa_\beta$	δ_0
α	0	γ	δ	$\sigma_\alpha \vdash \sigma_{\delta,\gamma}$	δ_0	$\sigma_{\delta,\gamma}$	σ_δ
0	β	0	δ	δ_0	κ_β	δ_0	$\sigma_\delta \vdash \kappa_\beta$
0	0	γ	δ	δ_0	δ_0	$\sigma_{\delta,\gamma}$	σ_δ
α	0	0	δ	σ_α	δ_0	δ_0	σ_δ
α	0	0	0	σ_α	δ_0	δ_0	δ_0
0	β	0	0	δ_0	κ_β	δ_0	δ_0

Proposition 8.2. *If A is a square r -dimensional matrix with identical positive entries α_j in the j -th row, then*

$$\mu_j = \sigma_{\alpha_1} \boxplus \sigma_{\alpha_2} \boxplus \dots \boxplus \sigma_{\alpha_r}$$

for each $j \in [r]$, and the μ_j coincide with μ and μ_0 .

Proof. Since the columns of A are identical, the functions $K_{i,j}$ are the same for all j 's. Denote them $L_i = K_{i,j}$, where $i, j \in [r]$. Moreover,

$$L_i(z) = \frac{b_i}{z - \sum_{j=1}^r L_j(z)} \quad \text{and} \quad \sum_{i=1}^r L_i(z) = \frac{\sum_{i=1}^r b_i}{z - \sum_{j=1}^r L_j(z)}$$

for any $i \in [r]$. Therefore, $\sum_{i=1}^r L_i(z)$ is the K-transform of the measure

$$\sigma_{\alpha_1} \boxplus \sigma_{\alpha_2} \boxplus \dots \boxplus \sigma_{\alpha_r}$$

and since $K_{\mu_j}(z) = \sum_{i=1}^r K_{i,j}(z) = \sum_{i=1}^r L_i(z)$, the proof for μ_j is completed. It is then easy to see that we get the same result for μ and μ_0 . \blacksquare

Proposition 8.3. *If A is a lower-triangular r -dimensional matrix with identical positive entries α_j in the j -th row below and on the main diagonal, then*

$$\mu_j = \sigma_{\alpha_j} \triangleright (\sigma_{\alpha_{j+1}} \triangleright (\dots \triangleright \sigma_{\alpha_r}) \dots)$$

for each $j \in [r]$. Moreover, $\mu_0 = \mu_1$ and μ is the convex linear combination of the μ_j as in Theorem 7.1.

Proof. As in the proof of Proposition 8.2, note that the $\mu_{i,j}$ do not depend on j and thus we can set $L_i = K_{i,j}$ for any $i \geq j$. If $i = r$, we have

$$L_r(z) = \frac{b_r}{z - L_r(z)},$$

using the continued fraction for the $K_{r,j}$ of Lemma 7.1. Therefore, $\mu_{r,j} = \sigma_{\alpha_r}$ for any $j \leq r$. Next, we have

$$L_k(z) = \frac{b_k}{z - \sum_{i=k}^r L_i(z)}$$

which leads to

$$L_k(z) = K_{\sigma_{\alpha_k}}(z - \sum_{i=k+1}^r L_i(z))$$

which gives the orthogonal decomposition of $\mu_{k,j}$,

$$\mu_{k,j} = \sigma_{\alpha_k} \vdash (\mu_{k+1,j} \uplus \dots \uplus \mu_{r,j})$$

for any $j \leq k < r$. Now, we claim that

$$\mu_{i,j} \uplus \dots \uplus \mu_{r,j} = \sigma_{\alpha_i} \triangleright (\sigma_{\alpha_{i+1}} \triangleright (\dots \triangleright \sigma_{\alpha_r}) \dots)$$

for any $1 \leq j \leq i \leq r$. Clearly, it holds for $i = r$ and any $j \leq r$ since we have already shown that $\mu_{r,j} = \sigma_{\alpha_r}$ for any $j \leq r$. Suppose now that this formula holds for $i > k$ and any $j \leq i$. We will show that it holds for $i = k$ and any $j \leq k$. Using the orthogonal decomposition of $\mu_{k,j}$ given above and the inductive assumption, we obtain

$$\mu_{k,j} = \sigma_{\alpha_k} \vdash (\sigma_{\alpha_{k+1}} \triangleright (\dots \triangleright \sigma_{\alpha_r}) \dots).$$

However, for any $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, we have a simple relation

$$(\mu \vdash \nu) \uplus \nu = \mu \triangleright \nu$$

which gives

$$\mu_{k,j} \uplus \dots \uplus \mu_{r,j} = \sigma_{\alpha_k} \triangleright (\sigma_{\alpha_{k+1}} \triangleright (\dots \triangleright \sigma_{\alpha_r}) \dots)$$

and the desired expression for μ_j . In a similar manner we obtain μ_0 and μ . \blacksquare

Example 8.1. If $\alpha = \delta \neq 0$ and $\beta = \gamma = 0$, then we can use Table 1 to obtain $\mu_0 = \sigma_{\alpha} \uplus \sigma_{\alpha}$, which is the arcsine law with Cauchy transform $G_{\mu_0}(z) = 1/\sqrt{z^2 - \alpha^2}$ whereas μ is the Wigner semicircle distribution σ_{α} . In turn, if $\alpha = \delta = 0$ and $\beta = \gamma \neq 0$, then $\mu_0 = \delta_0$, whereas $\mu = \sigma_{\beta}$.

9. WEIGHTED BINARY TREES AND CATALAN PATHS

In this Section we show how to express the limit distributions in terms of walks on weighted binary trees, or equivalently, in terms of weighted Catalan paths. The binary tree serves here as an example of the strongly matricially free Fock space.

The usual framework which gives a description of distributions in terms of walks on graphs is the following. Let $W(n)$ denote the set of root-to-root walks of length n on a rooted graph (\mathcal{G}, e) and let μ be the spectral distribution of (\mathcal{G}, e) , i.e. the distribution given by the moments of the adjacency matrix $A(\mathcal{G})$ in the state φ associated with the vector δ_e on the space of square integrable functions on the set $V(\mathcal{G})$ of the vertices of \mathcal{G} . Then the n -th moment of $A(\mathcal{G})$ in the state φ is equal to the cardinality of the set $W(n)$. In particular, it is well known that the moments of the Wigner semicircle distribution of variance 1 can be expressed in terms of walks on the half-line (\mathbb{T}_1, e) with the first vertex denoted by e and chosen as the root.

For many distributions we have to use a more general framework, in which the moments of these distributions are expressed in terms of root-to-root (random or, more generally, weighted) walks on some rooted graph, except that to each walk w on this graph we have to assign a real-valued weight $\xi(w)$. Then we can write

$$M_{\mu}(n) = \sum_{w \in W(n)} \xi(w)$$

for any $n \geq 1$, where μ is the considered distribution. In particular, we obtain the moments of ω_{α} for any $\alpha > 0$ by putting $\xi(w) = \alpha^n$, where $n = |w|$ is the length of w .

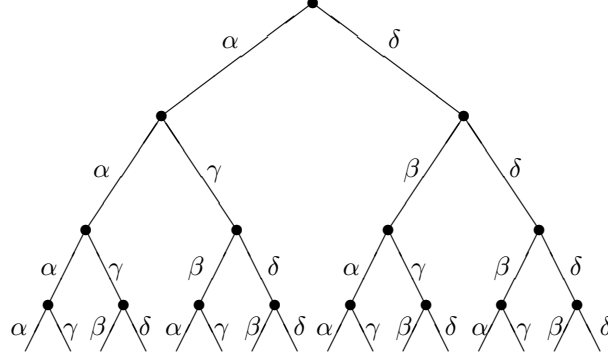


FIGURE 3. Binary tree with a matricial weight function

In the cases which are of interest to us, the weight function ξ is first defined on the set of edges $E(\mathcal{G})$ of \mathcal{G} and then is extended to $W = \bigcup_{n \geq 1} W(n)$ by multiplicativity. Namely, if we are given a mapping $\xi : E(\mathcal{G}) \rightarrow \mathbb{R}$, we set

$$\xi(w) = \xi(E_1)\xi(E_2) \dots \xi(E_n)$$

where $w = (E_1, E_2, \dots, E_n)$ and E_1, E_2, \dots, E_n are the edges of w (we choose to describe walks on graphs as sequences of edges). Such extension, by abuse of notation denoted also by ξ , will be called *multiplicative*.

For instance, it is easy to see that the moments of $\mu = \sigma_\alpha \boxplus \sigma_\delta$ can be expressed in this form. It is enough to take the free product of two half-lines, which is the binary tree (\mathbb{T}_2, e) with root e . Let us color this graph in the natural way, namely each edge which belongs to a copy of the first half-line is colored by 1 and each edge which belongs to a copy of the second half-line is colored by 2. Then the above formula holds for the moments of μ if we take $\xi(E) = \alpha$ whenever E is colored by 1 and $\xi(E) = \delta$ whenever E is colored by 2.

We will demonstrate below that we can express our distributions μ_0, μ_1, μ_2 in a similar form (in particular, we can use the binary tree), except that the weight function ξ will depend on all four parameters which appear in the matrix A . Before formulating the theorem, let us introduce special weight functions related to matricial freeness.

Definition 9.1. Let \mathbb{T}_r be a r -ary rooted tree with root e and let $A \in M_r(\mathbb{R})$. The weight function $\xi : E(\mathbb{T}_p) \rightarrow \mathbb{R}$ which assigns the entries of A to the edges of \mathbb{T}_r is called *matricial* if, for any pair of edges $E_1, E_2 \in E(\mathbb{T}_r)$, incident on the same vertex and such that E_1 is the ‘father’ of E_2 , the following implication holds:

$$\xi(E_1) = a_{i,j} \text{ for some } i, j \Rightarrow \xi(E_2) = a_{k,i} \text{ for some } k.$$

The unique multiplicative extension of this weight function to the set of all walks on \mathbb{T}_r will also be called *matricial*.

We specialize to $p = 2$ and the binary tree. Note that any matricial weight function ξ on the binary tree is uniquely determined (up to equivalence) by the set of those entries of the matrix A which are assigned to the set $\{E_1, E_2\}$ of two edges incident on the

root of tree, called the *initial weights*. In order to establish a connection with our limit distributions, we will assume, as in Section 8, that A is the ‘square root’ of $B = DU$ of Section 6. Then, in particular, the binary tree with the matricial weight function associated with A and the initial weights $\{\alpha, \delta\}$, shown in Figure 3, describes μ_0 as we show below.

Finally, recall after [1,12] that if (\mathcal{G}_1, e_1) and (\mathcal{G}_2, e_2) are two locally finite simple graphs and μ_1 and μ_2 are the associated spectral distributions, then the s-free product of \mathcal{G}_1 and \mathcal{G}_2 (in that order) can be interpreted as this half of the free product $\mathcal{G}_1 * \mathcal{G}_2$ which ‘begins’ (starting from the root) with a copy \mathcal{G}_1 . Moreover, the associated spectral distribution is given by $\mu_1 \boxplus \mu_2$. Let us add that a similar result holds in the multiplicative case: both the s-free multiplicative convolution and the associated s-free loop product of graphs were introduced in [13].

Below we shall use the s-free product of half-lines, which are (left and right) halves of the binary tree.

Theorem 9.1. *Let ξ_0, ξ_1, ξ_2 be the multiplicative matricial weight functions on the set of walks on \mathbb{T}_2 associated with matrix A and the initial weights $\{\alpha, \delta\}$, $\{\beta, \delta\}$ and $\{\alpha, \gamma\}$, respectively. Then*

$$(9.1) \quad M_{\mu_j}(n) = \sum_{w \in W(n)} \xi_j(w)$$

for $j = 0, 1, 2$ and any $n \in \mathbb{N}$, where $W(n)$ denotes the set of root-to-root walks on \mathbb{T}_2 of length n .

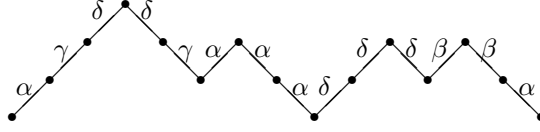
Proof. We need to translate the result of Corollary 8.1 to the language of graphs. It is well-known that the moments of σ_α can be interpreted in terms of walks on the half-line with the weight α assigned to each edge. The boolean compression T_t of σ_α changes only the weight assigned to the edge incident on the root, namely it multiplies it by $\sqrt{t} = \beta/\alpha$, which can be illustrated as

$$\begin{array}{ccc} \alpha & \alpha & \alpha & \alpha & \alpha \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \omega_\alpha & & & & \end{array} \xrightarrow{T_t} \begin{array}{ccc} \beta & \alpha & \alpha & \alpha & \alpha \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \omega_{\alpha,\beta} & & & & \end{array}$$

Now, the s-free additive convolutions of compressed semicircle distributions which appear in the decompositions of Corollary 8.1, namely

$$\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma} \quad \text{and} \quad \sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}$$

are the spectral distributions of the s-free products of these half-lines (taken in two different orders). These turn out to be the spectral distributions of the two halves of the binary tree \mathbb{T}_2 with the weight function defined by the initial set $\{\beta, \gamma\}$. In order to obtain μ_0 , we still need to apply $T_{1/t}$ and $T_{1/s}$, respectively, to the left and right halves of the tree, which amounts to changing the initial weights from β and γ , respectively, to α and δ . As a result, we obtain the weight function associated with A and the initial set $\{\alpha, \delta\}$. This proves the statement concerning μ_0 (the corresponding weight function on the binary tree is shown in Figure 3). The cases of μ_1 and μ_2 are very similar (at

FIGURE 4. A weighted Catalan path associated with A

the end, a boolean compression is applied to only one half of the binary tree). ■

Another geometric interpretation of Corollary 8.1 can be given in terms of Catalan paths. In order to define a Catalan path, we begin with a function $f : [2n] \rightarrow [n]$, such that $f(0) = f(2n)$ and $|f(i) - f(i-1)| = 1$ for any $1 \leq i \leq 2n$, and then we define a *Catalan path* as its unique extension $f : [0, 2n] \rightarrow [0, n]$ (by abuse of notation, denoted by the same symbol) obtained by connecting each $(i-1, f(i-1))$ with $(i, f(i))$ with a segment, where $1 \leq i \leq 2n$. Clearly, each Catalan path consists of segments of two types: ‘rises’ R_1, R_2, \dots, R_n , and ‘falls’ F_1, F_2, \dots, F_n . Moreover, to each ‘rise’ R_j there corresponds the closest ‘fall’ $F_{\tau(j)}$ lying to the right of R_j and on the same (vertical) level.

There is a natural mapping from the set $W(2n)$ of walks of length $2n$ on the binary tree and the set $C(n)$ of Catalan paths of length $2n$. In order to rephrase Theorem 8.1 in terms of Catalan paths, we need to take the sets of *weighted Catalan paths*, by which we understand pairs (f, ξ) , where f is a Catalan path and ξ is a real-valued weight function defined on the set of segments of f . The multiplicative formula

$$\xi(f) = \xi(R_1) \dots \xi(R_n) \xi(F_1) \dots \xi(F_n)$$

assigns the corresponding weight to f . Weighted Catalan paths of special type defined below allow us to rephrase Theorem 8.1.

Definition 9.2. A weighted Catalan path (f, ξ) of length $2n$ is called *matricial* if ξ assigns entries of $A \in M_p(\mathbb{R})$ to the segments of f in such a way that the following implications holds:

$$\xi(R_1) = a_{i,j} \text{ for some } i, j \Rightarrow \xi(R_2) = a_{k,i} \text{ for some } k,$$

$$\xi(F_1) = a_{i,j} \text{ for some } i, j \Rightarrow \xi(F_2) = a_{j,k} \text{ for some } k,$$

for any two consecutive ‘rises’ R_1, R_2 and two consecutive ‘falls’ F_1, F_2 , and the same weights are assigned to R_i and $F_{\tau(i)}$ for each $i \in [n]$.

We will consider below the set of matricially weighted Catalan paths of length $2n$ associated with the matrix A . In order to rephrase Theorem 9.1, using weighted Catalan paths, we need to restrict the set of weights which can be assigned to the first segment of each path (by analogy with trees, we call them *initial weights*). An example of a weighted Catalan path contributing to μ_0 is given in Figure 4.

Corollary 9.1. *Let $C_0(n)$, $C_1(n)$ and $C_2(n)$ be the sets of matricially weighted Catalan paths associated with A , with the initial weights $\{\alpha, \delta\}$, $\{\beta, \delta\}$ and $\{\gamma, \alpha\}$, respectively. Then*

$$M_{\mu_j}(2n) = \sum_{(f, \xi) \in C_j(n)} \xi(f)$$

for $j = 0, 1, 2$ and any $n \in \mathbb{N}$.

Proof. This is an easy consequence of Theorem 9.1 since there is a natural bijection between each $C_j(n)$ and the pair $(W(2n), \xi_j)$ for any n . ■

REFERENCES

- [1] L. Accardi, R. Lenczewski, R. Śaławata, Decompositions of the free product of graphs, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10** (2007), 303-334.
- [2] M. Anshelevich, Free Meixner states, *Commun. Math. Phys.* **276** (2007), 863-899.
- [3] D. Avitzour, Free products of C^* -algebras, *Trans. Amer. Math. Soc.* **271** (1982), 423-465.
- [4] Ph. Biane, Processes with free increments, *Math. Z.* **227** (1998), 143-174.
- [5] M. Bożejko, M. Leinert, R. Speicher, Convolution and limit theorems for conditionally free random variables, *Pacific J. Math.* **175** (1996), 357-388.
- [6] M. Bożejko, R. Speicher, ψ -independent and symmetrized white noises", in *Quantum Probability and Related Topics VI*, Ed. L. Accardi, World Scientific, Singapore, 1991, 170-186.
- [7] M. Bożejko, J. Wysocański, Remarks on t -transformations of measures and convolutions, *Ann. I.H. Poincaré- PR* **37**, 6 (2001), 737-761.
- [8] T. Cabanal-Duvillard, *Probabilités libres et calcul stochastique. Application aux grandes matrices aléatoires.*, Ph.D. Thesis, l'Université Pierre et Marie Curie, 1998.
- [9] T. Cabanal-Duvillard, V. Ionescu, Un théorème central limite pour des variables aléatoires non-commutatives, *C. R. Acad. Sci. Paris*, **325** (1997), Série I, 1117-1120.
- [10] W.M. Ching, Free products of von Neumann algebras, *Trans. Amer. Math. Soc.* **178** (1993), 147-163.
- [11] U. Franz, Multiplicative monotone convolutions, in *Quantum Probability*, Ed. M. Bożejko *et al*, Banach Center Publications, Vol. 73, p. 153-166, 2006.
- [12] R. Lenczewski, Decompositions of the free additive convolution, *J. Funct. Anal.* **246** (2007), 330-365.
- [13] R. Lenczewski, Operators related to subordination for free multiplicative convolutions, *Indiana Univ. Math. J.*, **57** (2008), 1055-1103.
- [14] R. Lenczewski, R. Śaławata, Discrete interpolation between monotone probability and free probability, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **9** (2006), 77-106.
- [15] R. Lenczewski, R. Śaławata, Noncommutative Brownian motions associated with Kesten distributions and related Poisson processes, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **11** (2008), to appear.
- [16] N. Muraki, Monotonic independence, monotonic central limit theorem and monotonic law of small numbers, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **4** (2001), 39-58.
- [17] N. Muraki, Monotonic convolution and monotonic Levy-Hincin formula, preprint, 2001.
- [18] A. Nica, Multi-variable subordination distribution for free additive convolution, arXiv:math.OA.0810.2571v1
- [19] R. Speicher, R. Woroudi, Boolean convolution, in *Free Probability Theory*, Ed. D. Voiculescu, 267-279, *Fields Inst. Commun.* Vol.12, AMS, 1997.
- [20] D. Voiculescu, Symmetries of some reduced free product C^* -algebras, *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Mathematics, Vol. 1132, Springer Verlag, 1985, pp. 556-588.
- [21] D. Voiculescu, Lectures on free probability theory, *Lectures on probability theory and statistics* (Saint-Flour, 1998), 279-349, Lecture Notes in Math. 1738, Springer, Berlin, 2000.

- [22] D. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104**(1991), 201-220.
- [23] D. Voiculescu , K. Dykema, A. Nica, *Free random variables*, CRM Monograph Series, No.1, A.M.S., Providence, 1992.
- [24] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, I, *Commun. Math. Phys.* **155** (1993), 71-92.
- [25] E. Wigner, On the distribution of the roots of certain symmetric matrices, *Ann. Math.* **67** (1958), 325-327.

ROMUALD LENCZEWSKI,
INSTYTUT MATEMATYKI I INFORMATYKI, POLITECHNIKA WROCŁAWSKA,
WYBRZEŻE WYSPIAŃSKIEGO 27, 50-370 WROCŁAW, POLAND

E-mail address: Romuald.Lenczewski@pwr.wroc.pl